Coexistence of Utilitarian Efficiency and False-name-proofness in Social Choice

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ABSTRACT

The class of Groves mechanisms has been attracting much attention in mechanism design literature due to two attractive characteristics: utilitarian efficiency (also called social welfare maximization) and dominant strategy incentive compatibility. However, when strategic agents can create multiple fake identities and reveal more than one preference under them, a refined characteristic called *false-name-proofness* is required. Utilitarian efficiency and false-name-proofness are incompatible in combinatorial auctions, if we also have *individual rationality* as a desired condition. However, although individual rationality is strongly desirable, if participation is mandatory due to social norms or reputations, a mechanism without individual rationality can be sustained.

In this paper we investigate the relationship between utilitarian efficiency and false-name-proofness in a social choice environment with monetary transfers. We show that in our modelization no mechanism simultaneously satisfies utilitarian efficiency, false-name-proofness, and individual rationality. Considering this fact, we ignore individual rationality and design various mechanisms that simultaneously satisfy the other two properties. We also compare our different mechanisms in terms of the distance to individual rationality. Finally we illustrate our mechanisms on a facility location problem.

Categories and Subject Descriptors

I.2.11 [**Artificial Intelligence**]: Distributed Artificial Intelligence—*Multiagent systems*; J.4 [**Social and Behavioral Sciences**]: Economics

Keywords

Mechanism design, False-name-proofness, Social welfare maximizer, Individual rationality

1. INTRODUCTION

Selecting one social outcome from a set of possible outcomes based on the revealed preferences of selfish agents is a fundamental problem in economics/social choice literatures. To achieve a social optimal outcome (e.g., welfare maximizing), a decision scheme known as a *social choice function* is needed that encourages agents to truthfully reveal their preferences. Unfortunately, as the Gibbard-Satterthwaite theorem argues [8, 16], it is impossible to get such a social choice function that satisfies some desired properties. Therefore, a natural approach is to implement a social choice function as a *mechanism* that consists of a social choice function and a well-designed monetary transfer scheme, i.e, a *transfer function*.

The development of the class of *Groves mechanisms* [10] is one of the great successes of mechanism design theory. Any Groves mechanism is *dominant strategy incentive compatible*, i.e. for every agent, a truthful revelation of her preference is always a best strategy for any given revealed preferences of the other agents. The outcome derived by a Groves mechanism is also *utilitarian efficient*, i.e. maximizes the sum of the gross utilities of agents. One of the most prominent results on Groves mechanisms is that under some natural conditions, they are the only mechanisms satisfying both of those properties [9]. Due to these advantages, the Groves mechanisms and their extensions have been studied in several mechanism design literatures.

In this paper we study social choice problems where agent participations are *mandatory*, perhaps reflecting the duties or responsibilities faced by members of a particular community. But the social choices affect the agents in the community even though they had the opportunity to avoid participating in the decision-making procedure. Let us consider the decision making by a development community for a new feature that will be included in a new release of an open source software, e.g., Linux. Due to their responsibilities for the development, the developers are expected to acquiesce to the community's new decision. Their participation is mandatory in a sense that they must spend time and effort on the new feature and the choice affects them, regardless of their participation in the decision-making procedure. In this case, a developer might have an incentive to create dummy developer accounts that support his own opinions, e.g., a different feature that involves less work (and thus a benefit) for his share of the implementation.

Such fraud is known as *false-name manipulation* in mechanism design literatures, especially in combinatorial auctions [6]. Robustness against false-name manipulations is called *false-name-proofness*, which is a refinement of dominant strategy incentive compatibility. It informally requires that for every agent, the truthful revelation of her preferences under her true account/identity is always the best strategy, for any given revealed preferences of other agents. Previous work has shown [23] that in combinatorial auc-

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tions, since no mechanism exists that satisfies this property, utilitarian efficiency, and *individual rationality*, requiring a truthful revelation is always weakly better than not participating. This negative result still seems valid even in our social choice context (formally shown in Section 3). Thus in this paper we ignore individual rationality and focus on false-name-proofness and utilitarian efficiency.

Violating individual rationality seems a big disadvantage for a mechanism, especially when the participation of agents is voluntary, since they might not have a strong incentive to participate in decision-making procedures. However, even in that case, it is still important to understand what level of efficiency can be achieved as well as false-name-proofness by violating individual rationality. Furthermore, in the situation described above where participation is mandatory, refusing to participate is not an option, but agents may benefit from false-name manipulations.

The main objective of this paper is to clarify the compatibility of false-name-proofness and utilitarian efficiency in a social choice environment. We will discuss whether false-name-proof and utilitarian efficient mechanisms exist, preferably by also satisfying individual rationality. To the best of our knowledge, besides single-item auctions and a few other simple settings [17], no false-name-proof and utilitarian efficient mechanisms have been discovered so far.

1.1 Our Results

We model false-name manipulations in social choice problems with monetary transfers. We first show an impossibility result on individual rationality; there exists no mechanism satisfying individual rationality with false-name-proofness and utilitarian efficiency under almost all preference domains. This result implies that the VCG mechanism is not false-name-proof in social choice problems.

We then focus on designing false-name-proof and utilitarian efficient mechanisms. We present two mechanisms that both satisfy these properties. They also support our approach to ignore individual rationality. Adhering to individual rationality in previous works on false-name-proof mechanisms [22, 23, 5, 12] caused a big loss of efficiency in social choices.

To illustrate their behaviors, we apply our two mechanisms to facility location problem on a straight line, a well-known problem in social choice theory. Since in the facility location problem the utilitarian efficient social choice function coincides with the median location, those mechanisms can be considered improvements of a quite negative result by Todo et al. [18], which states that without monetary transfer the leftmost location mechanism is essentially the only false-name-proof one. This leftmost mechanism may provide arbitrary small social welfare in our model.

1.2 Related Works

The Groves mechanisms [10] are sound frameworks to design utilitarian efficient and dominant strategy incentive compatible mechanisms. A special case of the Groves mechanisms is the one designed with Clarke's pivotal rule [4], which is also called the Vickrey-Clarke-Groves (VCG) mechanism [19, 4, 10]. Green and Laffont [9] proved that under natural assumptions Groves mechanisms are the only ones that satisfy utilitarian efficiency and dominant strategy incentive compatibility. Roberts [15] introduced a class of parameterized social choice functions called *affine maximizers*, which coincide with the class of dominant-strategy implementable ones in unrestricted domains and include the utilitarian efficient one as an extreme case.

The Groves mechanisms and their extensions are applied in various mechanism design fields. One of the most prominent fields is combinatorial auctions [6], in which the VCG mechanism is considered one of the benchmarks. In task scheduling to multiple processors [1, 20], VCG and affine maximizers are mostly referred as examples of dominant strategy incentive compatible mechanisms, even though they are not optimal in terms of minimizing the makespan, i.e., the greatest cost for an agent. Even for more general resource allocation problems [13, 11], the Groves mechanisms are applied when we need to consider agents' incentives. Garg et al. [7] introduced the idea of Groves mechanisms into supply chain management.

Yokoo et al. [22] initiated research on false-name-proof mechanism design and pointed out that the VCG mechanism is not false-name-proof in general combinatorial auction problems. Moreover, they provided an impossibility result, where there exists no mechanism that simultaneously satisfies utilitarian efficiency, false-name-proofness, and individual rationality for combinatorial auctions. Iwasaki et al. [12] focused on the worst-case efficiency of false-name-proof combinatorial auctions and provided a tight bound of $\frac{2}{m+1}$, where m is the number of objects. Conitzer [5], Wagman and Conitzer [21], and Todo et al. [18] discussed the effect of false-name manipulations in social choice problems, especially voting situations. Aziz et al. [2] and Penna et al. [14] extended the concept of false-name manipulations to cooperative games.

2. PRELIMINARIES

Let I be the set of agents that might be involved in the decision process. This set can be seen as the set of all agents in the world. Actually only subset $N \subseteq I$ of the agents is ultimately involved in the decision process (this subset is unknown before the decision process take place). Let X be the set of potential solutions that can be chosen by the community. Agent $i \in N$ gives his preferences over X through value function $u_i : X \to \mathbb{R}$. Let $U_i \subseteq \mathbb{R}^X$ be the set of possible preferences for agent i. For any $N \subseteq I$ let $U_N = \prod_{i \in N} U_i$ be the *preference domain* on N. Any $u \in U_N$ represents a preference profile on X for all agents belonging to N (if $N = \{1, \ldots, n\}$ then $u = (u_1, \ldots, u_n)$ where u_i represents the value function revealed by agent $i \in N$). One solution among X will be ultimately chosen based on preference profile u revealed by the agents of N. This choice is performed by social choice function $F : \bigcup_{N \subseteq I} U_N \to X$. In this paper we focus on the *social choice function* which chooses one of the solutions that maximize social welfare ($\forall N \subseteq I, \forall u \in U_N, F(u) \in \arg\max_{x \in X} \{\sum_{i \in N} u_i(x)\}$).

Rational agents may lie about their true preferences to obtain a better solution from social choice function F (with the highest value). To prevent such behavior, a system of money transfers is needed to force agents to reveal their true preferences. This quasi-linear assumption can be applied whenever money transfers are possible and whenever agents' value functions correspond to the amount of money they are willing to pay for a solution. To properly define these transfer functions, first we need to introduce notation u_{-i} to represent the restriction of preference profile $u \in U_N$ to the set of all agents of N, except agent i (this restricted preference profile belongs to $U_{N\setminus\{i\}}$). For any agent $i \in N$ *transfer function* $t_i : X \times \bigcup_{N' \subseteq I\setminus\{i\}} U_{N'} \to \mathbb{R}$ is such that if solution $x \in X$ is chosen by social choice function F when $u \in U_N$ is the revealed preference profile by the agents of $N \subseteq I$, then amount of money $t_i(x, u_{-i})$ is transferred from the decision maker to agent i. Let $t = \{t_i\}_{i \in I}$ be the set of transfer functions applicable to any agent who may be involved in the process (for any $i \in I, t_i$ represents a family of transfer functions for any subset $N \subseteq I$, which includes i). *Mechanism* (F, t) includes a social choice function (here the social welfare maximizer) and a set of transfer functions, which need to be designed to force agents to reveal their true preferences:

Definition 1. *Mechanism (F, t) is said to be incentive compatible if $\forall N \subseteq I, \forall u \in U_N, \forall i \in N, \forall u'_i \in U_i$ we have:*

$$u_i(F(u))+t_i(F(u), u_{-i}) \geq u_i(F(u'_i, u_{-i}))+t_i(F(u'_i, u_{-i}), u_{-i}).$$

This definition means that if u_i is the true value function of agent i and u_{-i} is the preference profile revealed by the other agents, then the transfer function is designed such that if agent i is rational then he prefers to reveal his true preferences instead of any other preference u'_i.

When the preference domain is *unrestricted* ($\forall i \in N, U_i = \mathbb{R}^X$) it was proved [9] that any incentive compatible mechanism belongs to the following family of mechanisms:

Definition 2. *[10] Mechanism (F, t) is called a Groves mechanism if for any $N = \{1, \ldots, n\} \subseteq I$ there are functions $h_1, \ldots, h_n$ where $\forall i \in N, h_i : U_{N\setminus\{i\}} \to \mathbb{R}$ is such that:*

$$t_i(F(u), u_{-i}) = \sum_{j \in N\setminus\{i\}} u_j(F(u)) + h_i(u_{-i}).$$

The property of incentive compatibility holds for any Groves mechanism not only when the domain is unrestricted but also for any preference domain. Set of functions $\{h_i\}_{i \in N}$ is the parameters that mechanism designers can use to confer properties to the Groves mechanism. One desirable property for a mechanism with transfers is the following:

Definition 3. *Mechanism (F, t) is individualy rational whenever $\forall N \subseteq I, \forall u \in U_N, \forall i \in N$ we have:*

$$u_i(F(u)) + t_i(F(u), u_{-i}) \geq u_i(F(u_{-i})). \qquad (1)$$

An individually rational mechanism has a property where no agent has an incentive to defect to it. This property interests agents who might hide from the mechanism. This property of individual rationality can be attained by the Groves mechanism by choosing the right set of functions $\{h_i\}_{i \in N}$ (like t_i, these functions rely on the set of involved agent $N \subseteq I$ that can be inferred from the parameters). For example, the *VCG mechanism* defines $\forall N \subseteq I, \forall u \subseteq U_N, \forall i \in N$ as function h_i for any $u_{-i} \in U_{-i}$ as

$$h_i(u_{-i}) = -\max_{x' \in X} \{ \sum_{j \in N\setminus\{i\}} u_j(x') \}.$$

This VCG mechanism has the property of individual rationality.

In some situations agents may have the possibility to create fake identities to manipulate social choice functions. These manipulations are much harder to prevent whenever the decision maker cannot see the difference between true and fake identities. To define these manipulations, we need to consider that I is a set of identities (instead of a set of agents) that belongs to an unknown subset of real agents. We assume that the size of I can be unbounded to model the fact that an agent can create as many fake identities as he wants to manipulate the mechanism. We still consider that ultimately a subset of identities $N \subseteq I$ will participate in the decision process, but now we consider that multiple identities of N can belong to the same real agent. Since the decision maker cannot distinguish between true and fake identities, the mechanism has to treat any of these identities as a possible true agent and choose a solution using the same social choice function introduced above. However to prevent fake identity manipulations, we need to consider any subset of identities of N a possible manipulation of a unique true agent. To define the manipulation of these fake identities we need to introduce notation u_{-B} for any $B \subseteq N$ to represent the restriction of preference profile $u \in U_N$ to all agents in N except agents belonging to B. The following definition corresponds to a mechanism's property to prevent such fake identity manipulations (or *false-name manipulation*):

Definition 4. *Mechanism (F, t) is said to be false-name-proof if $\forall N \subseteq I, \forall u \in U_N, \forall B \subseteq N, \forall i \in B$ and $\forall u'_i \in U_i$ we have:*

$$u'_i(x) + t_i(x, u_{-B}) \geq u'_i(F(u)) + \sum_{j \in B} t_j(F(u), u_{-j})$$

where $x = F(u'_i, u_{-B})$.

In this definition B is a subset of the identities that might belong to unique agent i with true preferences u'_i. If agent i revealed his true preferences and did not use any fake identities of B, then solution $F(u'_i, u_{-B})$ would have been chosen and agent i would have paid $t_i(x, u_{-B})$. The income for agent i from this possible manipulation is his true value for solution $F(u)$ plus the transfer function value sum of his identities (we consider that these transfers are assumed by the manipulating agent). Note that the possibility of false-name manipulation needs some assumptions about both social choice and transfer functions like anonymity (the social welfare maximizer is the only affine maximizer with this anonymity property).

This definition of false-name-proofness obviously generalizes the notion of incentive compatibility, which only considers the case where an agent uses one identity to trick the mechanism. Since incentive compatibility is one of the features of false-name-proofness, we need to focus on the set of Groves mechanisms in the unrestricted domain case. But not all Groves mechanisms are false-name-proof for any domain. Especially for the unrestricted domain case we know that VCGs are not always false-name-proof [23]. One question of interest is whether we can design a utilitarian efficient mechanism with the properties of false-name-proofness and individual rationality for a given domain.

3. IMPOSSIBILITY RESULT

In this section we show that when the considered social choice function corresponds to social welfare maximizer F, the properties of individual rationality and false-name-proofness described in the preliminaries are incompatible for a large set of preference domains. We first define the set of domains where this impossibility result applies:

DEFINITION 5. *Preference domain U_I is said to be symmetric whenever there exists $D \subseteq R^X$ such that for any $i \in I$ we have $U_i = D$.*

This assumption means that the sets of possible preferences of any identities are identical. This assumption does not seem so restrictive whenever the property of anonymity is needed. A second assumption about the domain is needed before stating our main result:

DEFINITION 6. *Preference domain U_I is said to be competitive whenever there exists $x, y \in X$ such that $\forall i \in N$ we can find $u_i^x, u_i^y \in U_i$ such that $\forall z \in X \backslash \{x\}, u_i^x(x) > u_i^x(z)$ and $\forall z \in X \backslash \{y\}, u_i^y(y) > u_i^y(z)$.*

The competitiveness assumption means that at least two solutions exist that might be strictly better than the other solutions for any agent. When the domain is symmetric this definition does not apply in a few rare cases (only if all agents agree about one best solution or all agents are indifferent to the solution chosen). The following result shows that there is little hope to find an individually rational, utilitarian efficient, and false-name-proof mechanism for most social choice problems:

PROPOSITION 1. *Whenever the preference domain is symmetric and competitive, false-name-proofness and individual rationality are incompatible for the social welfare maximizer.*

PROOF. Let us consider by contradiction that we can find t such that false-name-proofness and individual rationality hold. By symmetric assumption we know that $\exists D \subseteq \mathbb{R}^X$ such that $\forall i \in I, U_i = D$. By competitive assumption we know that $\exists f, g \in D$ such that $\forall z \in X \backslash \{x\}, f(x) > f(z)$ and $\forall z \in X \backslash \{y\}, g(y) > g(z)$. Let us first create a preference profile only using value functions f and g such that x is the solution chosen by social choice function F. Let $\alpha \in [0, 1]$ be the highest value such that $\exists z \in X \backslash \{x\}$,

$$\alpha f(x) + (1 - \alpha) g(x) = \alpha f(z) + (1 - \alpha) g(z).$$

Since $\forall z \in X \backslash \{x\}, f(x) > f(z)$, we have $1 > \alpha$ and $\exists p, q \in \mathbb{N}$ such that $1 > \frac{p}{q} > \alpha$. Let $\alpha' \in [0, 1]$ be the smallest value such that $\exists z \in X \backslash \{y\}$,

$$\alpha' f(y) + (1 - \alpha') g(y) = \alpha' f(z) + (1 - \alpha') g(z).$$

Since $\forall z \in X \backslash \{y\}, g(y) > g(z)$, we have $\alpha' > 0$ and $\exists p', q' \in \mathbb{N}$ such that $\alpha' > \frac{p'}{q'} > 0$. Now consider a set of q agents $N = \{\alpha_1, \ldots, \alpha_{p+1}, \beta_1, \ldots, \beta_{q-p-1}\}$. Define the value functions of agents in N by $\forall j \in \{\alpha_1, \ldots, \alpha_{p+1}\}, u_j = f$ and $\forall j \in \{\beta_1, \ldots, \beta_{q-p-1}\}, u_j = g$. The preference profile $u = (u_{\alpha_1}, \ldots, u_{\alpha_{p+1}}, u_{\beta_1}, \ldots, u_{\beta_{q-p-1}})$ belongs by assumption to U_N and might be a revealed preference profile. By the definitions of p and q we have $F(u) = x$.

Let $i \in \{\alpha_1, \ldots, \alpha_{p+1}\}$ and let $t_i(x, u_{-i}) = \gamma$ be the fixed value of the transfer for i in that situation. We show that the value of the transfer function needs to be strictly greater than γ for (F, t) to simultaneously be false-name-proof and individually rational. For that purpose we define $\epsilon = g(y) - g(x)$. Let also $k \in \mathbb{N}$ be the smallest value such that $[(kp + 1) - p]\epsilon > \gamma$ (since $\epsilon > 0$, such k exists). We construct a new preference profile that reflects one possible cheating opportunity for agent i. This new preference profile will be defined for a new subset of identities N' including N plus the set of fake identities used by i. Consider $N' = \{\alpha_1, \ldots, \alpha_{kp+1}, \beta_1, \ldots, \beta_{k(q-p)-1}\}$ where $\{\alpha_1, \ldots, \alpha_{p+1}\}$ and $\{\beta_1, \ldots, \beta_{q-p-1}\}$ are the same identities as in N, and $\{\alpha_{p+2}, \ldots, \alpha_{kp+1}\}$ and $\{\beta_{q-p}, \ldots, \beta_{k(q-p)-1}\}$ are the fake identities of i. The agents of N keep the same value functions (including i) and $\forall j \in \{\alpha_{p+1}, \ldots, \alpha_{kp+1}\}$, $u_j = f$ and $\forall j \in \{\beta_{q-p}, \ldots, \beta_{k(q-p)-1}\}, u_j = g$. Let $u' = (u^x_{\alpha_1}, \ldots, u^x_{\alpha_{kp+1}}, u^x_{\beta_1}, \ldots, u^x_{\beta_{k(q-p)-1}})$ denote this new preference profile (by assumption u' belongs to $U_{N'}$). By the definitions of p and q we have $F(u') = x$ (this manipulation does not change the solution chosen but this solution improved the value of the transfers for i). By false-name-proofness on N for i and his set of possible fake identities $N' \backslash N$, we need:

$$f(x) + t_i(x, u_{-i}) \geq f(x) + \sum_{l \in N' \backslash N} t_l(x, u'_{-l}) + t_i(x, u'_{-i}). \quad (2)$$

We presented u' as a possible manipulation of agent i but in some situations, u' can also represent a revealed preference profile for a set of true identities N' (in this situation, the identities of $N' \backslash N$ are no longer considered fake identities of i). Consider the case where for some $j \in \{\alpha_{p+2}, \ldots, \alpha_{kp+1}, i\}$ his true preferences are $u'_j = g$ (in that case, agent j did not reveal his true preferences). By the definitions of p and q, if agent j reveals his true value function u'_j, then the solution chosen and the value of the transfer function for him are the same as if he reveals u_j. Now in that case we define the opportunity of cheating for agent j and construct a new preference profile reflecting that opportunity. Let $k' \in \mathbb{N}$ be the smallest value such that $k'p' \geq kp + 2$ and $(q' - p')k' \geq (q - p)k$. This new preference profile will be defined for a new subset of identities N'' including N' and the set of fake identities used by j. Next consider $N'' = \{\alpha_1, \ldots, \alpha_{k'p'}, \beta_1, \ldots, \beta_{k'(q'-p')}\}$ where $\{\alpha_1, \ldots, \alpha_{kp+1}\}$ and $\{\beta_1, \ldots, \beta_{k(q-p)-1}\}$ are the same identities as in N', and $\{\alpha_{kp+2}, \ldots, \alpha_{k'p'}\}$ and $\{\beta_{k(q-p)}, \ldots, \beta_{k'(q'-p')}\}$ are the fake identities of j. The agents of N' keep the same value and $\forall k \in \{\alpha_{kp+2}, \ldots, \alpha_{k'p'}\}, u_k = f$ and $\forall k \in \{\beta_{k(q-p)}, \ldots, \beta_{k'(q'-p')}\}, u_k = g$. Let $u'' = (u_{\alpha_1}, \ldots, u_{\alpha_{k'p'}}, u_{\beta_1}, \ldots, u_{\beta_{k'(q'-p')}})$ denote the associated preference profile. By the definitions of p' and q' we have $F(u'') = y$.

By false-name-proofness for N' and j (we assumed that j's true value function is g) we have:

$$g(x) + t_j(x, u'_{-j}) \geq g(y) + \sum_{k \in N'' \backslash N'} t_k(y, u''_{-k}) + t_j(y, u''_{-j}).$$

By Lemma 2 (see appendix) we know that $\forall l \in N'' \backslash N' \cup \{i\}, t_l(y, u''_{-l}) \geq 0$, and so we have:

$$t_j(x, u'_{-j}) \geq g(y) - g(x) = \epsilon. \quad (3)$$

By Lemma 2 we also know that $\forall l \in \{\beta_{q-p}, \ldots, \beta_{k(q-p)-1}\}$, $t_l(x, u'_{-l}) \geq 0$, so from (2) and (3) we can state that

$$t_i(x, u'_{-j}) \geq \sum_{j \in \{\alpha_{p+2}, \ldots, \alpha_{kp+1}, i\}} t_j(x, u'_{-j}) \geq [kp + 1 - p]\epsilon > \gamma,$$

which constitute a contradiction. □

This proposition can be seen as a quite negative result in our false-name-proofness research. A natural approach to overcome such an impossibility result is to abandon one of the requirements of Proposition 1. Next we consider that the individually rational property can be violated and present some false-name-proof and utilitarian efficient mechanisms.

4. FALSE-NAME-PROOFNESS AND DISTANCE TO INDIVIDUAL RATIONALITY

In this section we present some utilitarian efficient mechanisms with the false-name-proofness property for any considered domain. Because the unrestricted domain is the preference domain that allows the most opportunities to cheat, we focus on it.

Strategy proofness is one of the main features needed to obtain false-name-proofness. Since we know that for the unrestricted domain, we can restrict our research to the family of Groves mechanisms to obtain false-name-proofness [9]. We first show a necessary and sufficient condition on the functions defining the Groves mechanisms to obtain false-name-proofness for the unrestricted domain:

LEMMA 1. *When the domain is unrestricted, false-name-proofness for Groves mechanisms is equivalent to the fact that $\forall N \subseteq I, \forall u \in U_N, \forall B \subseteq N$, and $\forall i \in B$, and so the following inequality holds:*

$$h_i(u_{-B}) - \sum_{j \in B} h_j(u_{-j}) \geq (|B|-1) \max_{x' \in X} \{\sum_{j \in N} u_j(x')\}.$$

PROOF. False-name-proofness applied to the Groves mechanisms means that $\forall N \subseteq I, \forall u \in U_N, \forall B \subseteq N, \forall i \in B$ and $\forall u'_i \in U_i$, and we have:

$$u'_i(x) + \sum_{j \in N \setminus B} u_j(x) + h_i(u_{-B}) \geq u'_i(y) + \beta + \sum_{j \in B} h_j(u_{-j}),$$

where $x = F(u)$ and $y = F(u_{-B}, u'_i)$ and

$$\beta = |B| \sum_{j \in N \setminus B} u_j(y) + (|B|-1) \sum_{j \in B} u_j(y).$$

If the previous inequalities hold for $u'_i(x) - u'_i(y) = \sum_{j \in N \setminus B} u_j(y) - \sum_{j \in N \setminus B} u_j(x)$, then it holds for any $u''_i \in U_i$ such that $u''_i(x) - u''_i(y) \geq \sum_{j \in N \setminus B} u_j(y) - \sum_{j \in N \setminus B} u_j(x)$. The false-name-proofness for the Groves mechanisms is equivalent to the fact that $\forall N \subseteq I, \forall u \in U_N, \forall B \subseteq N$, and $\forall i \in B$, and the following inequality holds:

$$h_i(u_{-B}) \geq (|B|-1) \sum_{j \in N} u_j(y) + \sum_{j \in B} h_j(u_{-j}).$$

By the definition of F we can replace the term $\sum_{j \in B} u_j(y)$ by $\max_{x' \in X} \{\sum_{j \in N} u_j(x')\}$ to obtain our desired property. □

This property is convenient to show that a given Groves mechanism is false-name-proof. We can now present our first false-name-proofness mechanism:

PROPOSITION 2. *A Groves mechanism defined for any $N \subseteq I$, any $u \in U_N$, and any $i \in N$ by the following function*

$$h_i(u_{-i}) = - \sum_{j \in N \setminus \{i\}} \max_{x' \in X} \{u_j(x')\}$$

is false-name-proof for any domain.

PROOF. Consider $N \subseteq I, u \in U_N, B \subseteq N$ and $i \in B$. By the definition of h_j for $j \in B$, we have

$$\begin{aligned} & h_i(u_{-B}) - \sum_{j \in B} h_j(u_{-j}) \\ = & \sum_{j \in B} \sum_{k \in N \setminus \{j\}} \max_{x' \in X} \{u_k(x')\} - \sum_{j \in N \setminus B} \max_{x' \in X} \{u_j(x')\} \\ = & \sum_{j \in B \setminus \{i\}} \sum_{k \in N \setminus \{j\}} \max_{x' \in X} \{u_k(x')\} + \sum_{j \in B \setminus \{i\}} \max_{x' \in X} \{u_j(x')\} \\ = & \sum_{j \in B \setminus \{i\}} \sum_{k \in N} \max_{x' \in X} \{u_k(x')\}. \end{aligned}$$

It is obvious that for any possible X we have:

$$\sum_{k \in N} \max_{x' \in X} \{u_k(x')\} \geq \max_{x' \in X} \{\sum_{k \in N} u_k(x')\}.$$

From the previous equations we can state that:

$$h_i(u_{-B}) - \sum_{j \in B} h_j(u_{-j}) \geq (|B|-1) \max_{x' \in X} \{\sum_{k \in N} u_k(x')\}.$$

Finally from Lemma 1, we know that false-name-proofness holds for the unrestricted domain and consequently for any domain. □

We can reformulate the transfer function of this mechanism for preference profile $u \in U_N$ and agent $i \in N$ as

$$t_i(F(u), u_{-i}) = \sum_{j \in N \setminus \{i\}} [u_j(F(u)) - \max_{x' \in X} \{u_j(x')\}].$$

This transfer function corresponds to the sum of the other agents of their value gaps between their best solution and the solution chosen by the social choice function. It is obvious that the payments for this mechanism can be very large when the situation involves many participants.

Even if we have abandoned the property of individual rationality, we can still consider that it needs to be violated as little as possible. From that perspective we can define the distance for a mechanism to the property of individual rationality as follows:

DEFINITION 7. *The distance of mechanism (F, t) to individual rationality is defined for a given $N \subseteq I$ and a given $u \in U_N$ by the following function:*

$$d(u) = \max_{i \in N} \{d_i(u)\},$$

where for any $i \in N$

$$d_i(u) = u_i(F(u_{-i})) - u_i(F(u)) - t_i(F(u), u_{-i}).$$

For a given preference profile, function d_i defines the degree of the violation of the individual rationality for agent i and d represents the worst degree of the violation for an agent of N. We want to obtain a mechanism with as short a distance to individual rationality as possible for any preference profile. To define the worst case value of this distance for a mechanism, we use the following function:

$$\Delta(u) = \max_{j \in N} \left\{ \max_{x' \in X} \{u_j(x')\} - \min_{x'' \in X} \{u_j(x'')\} \right\},$$

as defined for any $N \subseteq I$ and any $u \in U_N$. For a given preference profile u, this value represents the greatest gap between the best and the worst solutions for an agent of N. Based on the previous definition, the following proposition presents the worst case distance to individual rationality for our first mechanism:

PROPOSITION 3. *For the mechanism of Proposition 2, the distance to individual rationality for a given $N \subseteq I$ and a given $u \in U_N$ never exceeds $(n-1)\Delta(u)$. Furthermore in the unrestricted domain case, we can find preference profiles where this bound is attained.*

PROOF. It is obvious for this mechanism that $\forall N \subseteq I, \forall u \in U_N$ and any $i \in N$, and so we have:

$$d_i(u) \leq (n-1)\Delta(u_{-i}) \leq (n-1)\Delta(u).$$

By the definition of d, we have

$$\max_{i \in N} \{d_i(u)\} \leq (n-1)\Delta(u).$$

Let us now construct preference profile $u \in U_N$ for $N \subseteq I$ such that this bound is attained. We consider a situation where $X = \{x_1, \ldots, x_n\}$, a value $v > 0$, and the preference profile $u \in U_N$ such that $\forall j \in N \backslash \{i\}, u_j(x_j) = v$, $\forall k \in N \backslash \{j\}, u_j(x_k) = 0$, and $\forall j \in N, u_i(x_j) = 0$. The solution chosen must belong to $\{x_j\}_{j \in N \backslash \{i\}}$. Whatever solution is chosen, we have $d_i(u_{-i}) = (n-1)v$. Since in that case $\Delta(u) = v$, we have:

$$d_i(u) = (n-1)\Delta(u),$$

and as a direct consequence, $d(u) = (n-1)\Delta(u)$. □

Unfortunately, this worst case distance grows not only with the greatest gap of value between two solutions but also with the number of agents involved in the mechanism.

Considering this negative aspect of the previous mechanism we present a second false-name-proof mechanism:

PROPOSITION 4. *A Groves mechanism defined for any $N \subseteq I$, for any $u \in U_N$, and for any $i \in N$ by the following function*

$$h_i(u_{-i}) = -\max_{x' \in X} \{ \sum_{j \in N \backslash \{i\}} u_j(x') \} - \Delta(u_{-i})$$

is false-name-proof for any domain.

PROOF. Consider $N \subseteq I, u \in U_N, B \subseteq N$, and $i \in B$. By the definition of h_j for $j \in B$, we have

$$h_i(u_{-B}) - \sum_{j \in B} h_j(u_{-j}) = \alpha + \sum_{j \in B} \Delta(u_{-j}) - \Delta(u_{-B}), \quad (4)$$

where

$$\alpha = \sum_{j \in B} \max_{x' \in X} \{ \sum_{k \in N \backslash \{j\}} u_k(x') \} - \max_{x' \in X} \{ \sum_{j \in N \backslash B} u_j(x') \}.$$

By the definition of Δ we can state that

$$\sum_{j \in B} \Delta(u_{-j}) - \Delta(u_{-B}) \geq (|B|-1)\Delta(u). \quad (5)$$

Let $x^* \in \arg\max_{x' \in X} \{\sum_{j \in N \backslash B} u_j(x')\}$. We know that $\max_{x' \in X} \{ \sum_{j \in N \backslash \{i\}} u_j(x') \} - \sum_{j \in N \backslash B} u_j(x^*) \geq \sum_{j \in B \backslash \{i\}} u_j(x^*)$, and so we can state that

$$\alpha \geq \sum_{j \in B \backslash \{i\}} \beta(j), \quad (6)$$

where for any $j \in B \backslash \{i\}$

$$\beta(j) = \max_{x' \in X} \{ \sum_{k \in N \backslash \{j\}} u_k(x') \} + u_j(x^*).$$

Since $\forall j \in B \backslash \{i\}$, we have $u_j(x^*) + G(u) \geq \max_{x' \in X} \{u_j(x')\}$, and it is obvious that for any X we have:

$$\beta(j) + u_j(x^*) + \Delta(u) \geq \max_{x' \in X} \{ \sum_{k \in N} u_k(x') \}. \quad (7)$$

By summing (7) for any $j \in B \backslash \{i\}$ we get:

$$\sum_{j \in B \backslash \{i\}} \beta(j) + (|B|-1)\Delta(u) \geq (|B|-1) \max_{x' \in X} \{ \sum_{k \in N} u_k(x') \}. \quad (8)$$

By summing (5), (6), and (8) and using (4) we obtain:

$$h_i(u_{-B}) - \sum_{j \in B} h_j(u_{-j}) \geq (|B|-1) \max_{x' \in X} \{ \sum_{k \in N} u_k(x') \}.$$

Finally from Lemma 1 we know that false-name-proofness holds for the unrestricted domain and consequently for any domain. □

The following proposition shows that this mechanism has a good property in terms of the distance to the individual compared to the previous mechanism:

PROPOSITION 5. *For the mechanism of Proposition 4, the distance to individual rationality for a given $N \subseteq I$ and a given $u \in U_N$ is never higher than $\Delta(u)$.*

PROOF. Let $N \subseteq I$ and $u \in U_N$. Since $F(u) \in \arg\max_{x' \in X} \{\sum_{j \in N} u_j(x')\}$, we know that for any $i \in N$ we have

$$\begin{aligned} d_i(u) &= \sum_{j \in N} u_j(F(u_{-i})) - \sum_{j \in N} u_j(F(u)) + \Delta(u_{-i}) \\ &\leq \Delta(u_{-i}). \end{aligned}$$

Since $\Delta(u) \geq \Delta(u_{-i})$ and by the definition of d, we get:

$$d(u) = \max_{j \in N} \{d_j(u)\} \leq \Delta(u).$$

□

This mechanism is much better than the mechanism proposed in Proposition 2 in terms of the worst case distance to individual rationality. Nevertheless we can still find some preference profiles where individual rationality is not violated by the mechanism of Proposition 2, but it is violated by the mechanism of Proposition 4. For example, consider preference profile $u \in U_N$ where $\exists x \in X$ such that $\forall j \in N$ and $\forall z \in X \backslash \{x\}$, we have $u_j(x) > u_j(z)$. In that case for the mechanism of Proposition 4 we have $\forall j \in N, d_j(u) = \Delta_j(u_j) > 0$ since $F(u) = F(u_{-i})$. On the other hand for the mechanism of Proposition 2 we have $d_j(u) = 0$. We conclude that neither of the two mechanisms are on any occasion better than the other.

5. FALSE-NAME-PROOFNESS AND FACILITY LOCATION PROBLEM

We illustrate in this section our impossibility result and our false-name-proof mechanisms on a well-known problem in social choice theory. This facility location problem must find the best place to install a facility on a straight line according to the agent positions on that segment. We consider segment $[a, b]$ that represents a street where the agents live. Position $l_i \in [a, b]$ of an agent on the segment defines his value function. We assume that for any $N \subseteq I$ the set of positions is ordered by increasing order (i.e., $\forall i \in N \backslash \{n\}, l_i \leq$

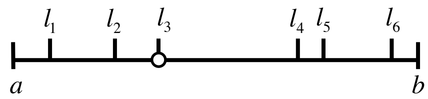

Figure 1: Instance of facilty location problem

l_{i+1}). We want to install the facility at location $d \in [a, b]$ to minimize the sum of the distances between d and position l_i of any agent $i \in N$. So we search for d, which maximizes $-\sum_{j \in N} |d - l_j|$. For any $j \in N, u_j(d) = -|d - l_j|$ defines the value function of agent j and the social choice function corresponds to the social welfare maximizer [3]. The solution chosen in that case is the median value of $\{l_j\}_{j \in N}$ ($l_{\frac{n}{2}}$ if $|N|$ is even and $l_{\frac{n+1}{2}}$ if $|N|$ is odd). Fig. 1 represents an instance of this problem for subset N of six agents. Based on the median rule, the location ultimately chosen in this example is l_3.

Since this domain is single peaked, consequently it is incentive compatible without money transfers. However this mechanism is not false-name-proof. Indeed, based on the example in Fig. 1, if agent 4 has the opportunity to create one fake identity and to declare identical position l_4 for it, then the position chosen for the facility changes to l_4. In that case the utility of agent i is strictly increased, and the property of false-name-proofness is violated. To obtain false-name-proofness in that problem, we need to consider money transfers.

The domain of preferences for this problem is symmetric because the position of any agent can be any point in $[a, b]$. If $a < b$, and then at least two different preference profiles exist such that one solution is strictly better than the other (for example, the preference profiles associated with positions a and b). According to Proposition 1, we know that in this restrictive domain we cannot obtain a false-name-proof and individually rational mechanism for the median rule.

We now describe the system of payments associated with the two false-name-proof mechanisms presented in Section 4. For the mechanism of Proposition 2 we need to define for any $u \in U_N$ and for any $j \in N$ the value of $u_j(F(u)) - \max_{x' \in X} \{u_j(x')\}$. Since the best solution for an agent is obviously his position, this value is equal to minus the distance between the facility's final location and the position of agent j. If $d \in [a, b]$ is the median value of $\{l_j\}_{j \in N}$, then the payment for agent $i \in N$ is $\sum_{j \in N \setminus \{i\}} |l_j - d|$. For the example of Fig. 1, the payment for agent 4 when he does not cheat is $\sum_{j \in N \setminus \{4\}} |l_j - l_3|$ and his utility for the chosen solution is $\sum_{j \in N} |l_j - l_3|$. If agent 4 uses one fake identity to cheat, as described above, then he needs to pay $\sum_{j \in N} |l_j - l_4|$ for each of her identities. Since l_3 is the median value of $\{l_j\}_{j \in N}$, we have $-\sum_{j \in N} |l_j - l_3| \geq -2 \sum_{j \in N} |l_j - l_4|$, and agent 4 has no incentive to perform this fraud.

For the mechanism of Proposition 4 we need to define for any $u \in U_N$ and any $i \in N$ the value of $\Delta(u_{-i})$. This value corresponds to minus the largest distance between a point in $\{l_j\}_{j \in N \setminus \{i\}}$ and extreme points a or b. The others terms of the transfer function correspond to the VCG payments, and we describe them in Lemma 3 of the appendix. The value of the payment for agent $i \in N$ is equal to the largest distance between a point in $\{l_j\}_{j \in N \setminus \{i\}}$ and a or b, plus, if $|N|$ is even and $i \in \{1, \ldots, \frac{n}{2}\}$, the distance between $l_{\frac{n}{2}}$ and $l_{\frac{n}{2}+1}$. For the example of Fig. 1, the payment for agent 4 when he does not cheat is $|a - l_6|$, and his utility for the chosen solution is $-|l_4 - l_3| - |a - l_6|$. If agent 4 uses a fake identity, as described above, he needs to pay $|a - l_6|$ for each of her identities. Since we have $-|l_4 - l_3| - |a - l_6| \geq -2|a - l_6|$, agent 4 has no incentive to perform this fraud.

6. CONCLUSION

In this article we showed that for the social welfare maximizer in a quasi-linear environment, individual rationality and false-name-proofness are incompatible for a wide class of social choice problems including the facility location problem. False-name-proofness can be attained when the requirement of individual rationality is abandoned by exhibiting two false-name-proof mechanisms. For both mechanisms, we studied their closeness to individual rationality and concluded that neither outperforms the other in any circumstances. Finally we illustrated how payments can be calculated for our mechanisms in the facility location problem on a line.

This work rises many questions about the possibility of designing false-name-proof mechanisms that are as close as possible to individual rationality. Is it possible to find a false-name-proof mechanism such that the distance to individual rationality that we mentioned is lower than both of our mechanisms? This concern can focus on restricted domains with favorable properties. Finally interest might also exist in relaxing another requirement of our impossibility result, for example, considering that the creation of an unrestricted number of fake identities is not possible for some reason.

7. ACKNOWLEDGMENTS

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APPENDIX

LEMMA 2. *Whenever preference domains are symmetric ($\exists D \subseteq \mathbb{R}^X$ such that $\forall i \in N, U_i = D$) and mechanism (F, t) is simultaneously individually rational and false-name-proof, then for any $x \in X$ such that $\exists f \in D$ with $\forall z \in X\backslash\{x\}, f(x) > f(z)$ we have $\forall N \subseteq I, \forall u \in U_N$ such that $F(u) = x, \forall i \in N, t_i(x, u_{-i}) \geq 0$.*

PROOF. By contradiction, assume that $\exists N \subseteq I, \exists u \in U_N$ such that $F(u) = x$ and $t_i(x, u_{-i}) < 0$ for some $i \in N$. Let us now consider one opportunity of cheating for agent i using set of fake identities $N' = \{\alpha_1, \ldots, \alpha_k\} \subseteq I\backslash\{i\}$. All these identities reveal the same value function f ($\forall j \in N', u'_j = f$). Also consider that agent i reveals $u'_i = f$ instead of his true value function u_i (except if $u_i = f$ but the result in that case is the same). Let $u' = (u_{-i}, u'_i, u'_{\alpha_1}, \ldots, u'_{\alpha_k})$ denote the new preference profile (by assumption $u' \in U_{N\cup N'}$). Size $k \in \mathbb{N}$ of N' corresponds to the minimum value such that

$$\arg\max_{x' \in X} \{ \sum_{j \in N\backslash\{i\}} u_j(x') + \sum_{j \in N'} u_j(x')\} = \{x\}.$$

(k is finite since $\forall j \in N', u_j = f$ and $\forall z \in X\backslash\{x\}, f(x) > f(z)$). In that case by cheating, agent i does not change the solution chosen by the social choice function ($F(u') = x$), but we show that the money transfers for him and his fake identities must be greater than or equal to zero.

Preference profile u' can also represent a situation where all the identities of $N \cup N'$ are true and reveal their true preferences. In that case, individual rationality needs to hold and for any $j \in N' \cup \{i\}$ we must have $u'_j(x) + t_j(x, u'_{-j}) \geq u'_j(x)$ (by the definition of k we have $\forall j \in N' \cup \{i\}, F(u') = F(u'_{-j}) = x$). For any $j \in N' \cup \{i\}$ we have:

$$t_j(x, u'_{-j}) \geq 0. \tag{9}$$

On the other hand by false-name-proofness for N and i, the following inequality must hold:

$$u_i(x) + t_i(x, u_{-i}) \geq u_i(x) + \sum_{j \in N' \cup \{i\}} t_i(x, u'_{-j}). \tag{10}$$

From (9) and (10) we obtain $t_i(x, u_{-i}) \geq 0$, which constitutes a contradiction. □

LEMMA 3. *The values of the transfer function of the VCG mechanism for the facility location problem on a line are 0 when the size of N is odd. Furthermore if the size of N is even, then the value of the transfer function for an agent in $\{1, \ldots, \frac{n}{2}\}$ is $l_{\frac{n}{2}} - l_{\frac{n}{2}+1}$ and 0 for the other agents.*

PROOF. Let $d \in [a, b]$ be the median value of $\{l_j\}_{j \in N}$, and let $d_i \in [a, b]$ be the median value of $\{l_j\}_{j \in N\backslash\{i\}}$. For the facility location problem on a line, the value of the transfer function for agent $i \in N$ corresponds to

$$\sum_{j \in N\backslash\{i\}} [|l_j - d_i| - |l_j - d|].$$

If $|N|$ is odd, then $d = l_{\frac{n+1}{2}}$. If i belongs to $\{1, \ldots, \frac{n-1}{2}\}$, then $d_i = d$ and the value of the transfer function is 0. If i belongs to $\{\frac{n+1}{2}, \ldots, n\}$, then $d_i = l_{\frac{n-1}{2}}$. For all $j \in \{1, \ldots, \frac{n-1}{2}\}$ we have $|l_j - d_i| - |l_j - d| = d_i - d$, and for all $\{\frac{n+1}{2}, \ldots, n\}\backslash\{i\}$ we have $|l_j - d_i| - |l_j - d| = d - d_i$. The value of the transfer function is also 0.

If $|N|$ is even, then $d = l_{\frac{n}{2}}$. If i belongs to $\{\frac{n}{2} + 1, \ldots, n\}$, then $d = d_i$ and the value of the transfer function is 0. Finally if i belongs to $\{1, \ldots, \frac{n}{2}\}$ then $d_i = l_{\frac{n}{2}+1}$. For all $j \in \{\frac{n}{2} + 1, \ldots, n\}\backslash\{i\}$ we have $|l_j - d_i| - |l_j - d| = d_i - d$ and for all $\{1, \ldots, \frac{n}{2}\}$ we have $|l_j - d_i| - |l_j - d| = d - d_i$. So the value of the transfer function is $d - d_i = l_{\frac{n}{2}} - l_{\frac{n}{2}+1}$. □