# Minimal Extending Sets in Tournaments

# (Extended Abstract)

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## ABSTRACT

In 2011, Brandt proposed a new tournament solution called the *minimal extending set* ($ME$). It was conjectured that $ME$ satisfies a large number of desirable properties. In this paper, we non-constructively show that $ME$ fails to satisfy most of these properties. However, no *concrete* examples of these violations are known and it appears that $ME$ satisfies these properties for all practical purposes. This casts doubt on the axiomatic method.

## Categories and Subject Descriptors

F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity; J,4 [**Social and behavioral sciences**]: Economics

## General Terms

Theory, Economics, Algorithms

## Keywords

Tournament solutions, Banks set, Minimal extending set

## 1. INTRODUCTION

Many problems in multiagent decision making can be addressed using tournament solutions. Examples of well-studied tournament solutions are the Copeland set, the Banks set, or the minimal covering set [4]. A common benchmark for tournament solutions is which desirable properties they satisfy (see e.g., [4], for an overview of such properties).

In 2011, Brandt [1] proposed a new tournament solution called $ME$ and an associated graph-theoretic conjecture. If the conjecture had held, $ME$ would have satisfied virtually all desirable properties that are usually considered in the literature on tournament solutions. In 2013, however, the existence of a counter-example with about $10^{136}$ alternatives was shown. The proof is non-constructive and uses the probabilistic method [3].

This left open which of the properties are actually satisfied by $ME$. In this paper, we resolve these open questions. Using the counter-example by Brandt et al. [3] we show that $ME$ fails to satisfy most properties (such as monotonicity and stability) while it does satisfy a stronger version of idempotency, irregularity, and membership in the Banks set. We also prove that computing $ME$ is NP-hard.

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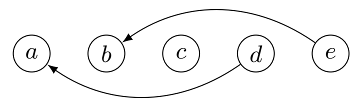


**Figure 1: In this tournament, $ME(T) = \{a,b,d\}$ whereas $BA(T) = \{a,b,c,d\}$. An edge from $a$ to $b$ corresponds to $a \succ b$. Omitted edges point rightwards.**

## 2. PRELIMINARIES

A *tournament* $T$ is a pair $(A, \succ)$, where $A$ is a set of alternatives and $\succ$ is an asymmetric and complete (and thus irreflexive) binary relation on $A$, usually referred to as the *dominance relation*. The dominance relation can be extended to sets of alternatives by writing $a \succ B$ when $a \succ b$ for all $b \in B$. Let $\mathcal{B}_T(a)$ denote the set of all subsets $B \subseteq A$ such that $\succ|_B$ is transitive and $a \succ B \setminus \{a\}$.

A *tournament solution* is a function that maps a tournament to a nonempty subset of its alternatives. The *Banks set* $BA$ chooses maximal elements of inclusion-maximal transitive subtournaments, i.e.,

$$BA(T) = \{a \in A : \exists B \in \mathcal{B}_T(a) \text{ such that } \nexists b\colon b \succ B\}.$$

A subset of alternatives $B \subseteq A$ is called *$S$-stable* for tournament solution $S$ if $a \notin S(B \cup \{a\})$ for all $a \in A \setminus B$. In particular, we refer to $BA$-stable sets as *extending sets*. The union of all inclusion minimal extending sets defines the tournament solution $ME$ [1], i.e.,

$$ME(T) = \bigcup\{B \text{ is } BA\text{-stable} : \forall C \subsetneq B : C \text{ is not } BA\text{-stable}\}.$$

An example of a tournament where $BA$ and $ME$ differ is given in Figure 1.

## 3. PROPERTIES OF ME

### *Dominance-based properties.*

First, we consider two properties that are based on the dominance relation. *Monotonicity* prescribes that a chosen alternative should still be chosen if it is reinforced. The second property, *independence of unchosen alternatives*, states that the choice set should be unaffected by changes in the dominance relation between unchosen alternatives.

Theorem 1. *ME satisfies neither monotonicity nor independence of unchosen alternatives.*

### *Choice-theoretic properties.*

An important class of properties concern the consistency of choice and relate choices from different subtournaments of the same tournament to each other. A relatively strong property of this type is *stability*, which requires that a set is chosen from two different sets of alternatives if and only if it is chosen from the union of these sets [2]. A tournament solution $S$ is *stable* if for all $(A, \succ)$, $B, C \subseteq A$, and $X \subseteq B \cap C$,

$$X = S(B) = S(C) \quad \text{if and only if} \quad X = S(B \cup C).$$

Stability can be factorized into conditions $\widehat{\alpha}$ and $\widehat{\gamma}$ by considering each implication in the above equivalence separately. A tournament solution $S$ satisfies $\widehat{\alpha}$, if for all sets of alternatives $A, B$, and $X$ with $X \subseteq A \cap B$, $X = S(A \cup B)$ implies $X = S(A) = S(B)$. A tournament solution $S$ satisfies $\widehat{\gamma}$, if for all sets of alternatives $A, B$, $S(A) = S(B)$ implies $S(A \cup B) = S(A) = S(B)$.

For a finer analysis, we split $\widehat{\alpha}$ and $\widehat{\gamma}$ into two conditions [2, Remark 1]. A tournament solution $S$ satisfies $\widehat{\alpha}_{\subseteq}$ (or $\widehat{\alpha}_{\supseteq}$) if $S(A) \subseteq B \subseteq A$ implies $S(B) \subseteq S(A)$ (or $S(B) \supseteq S(A)$). Similarly, a tournament solution $S$ satisfies $\widehat{\gamma}_{\subseteq}$ (or $\widehat{\gamma}_{\supseteq}$) if for all $A, B$, and $X$, it holds that $X = S(A) = S(B)$ implies $X \subseteq S(A \cup B)$ (or $X \supseteq S(A \cup B)$).

In Figure 2, the logical relations between the different properties are depicted.

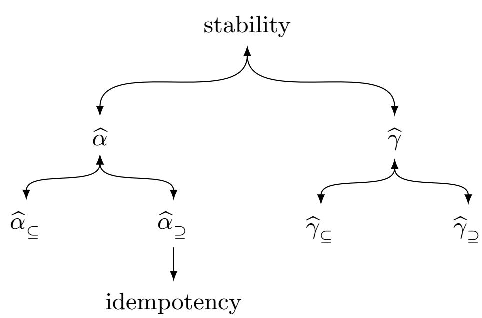


**Figure 2: Implications of stability properties.**

Theorem 2. *ME satisfies $\widehat{\alpha}_{\supseteq}$ but neither $\widehat{\alpha}_{\subseteq}$ nor $\widehat{\gamma}_{\supseteq}$.*

As a consequence, *ME* is not stable but satisfies idempotency. It is still open whether *ME* satisfies $\widehat{\gamma}_{\subseteq}$.

### *Relationships to other tournament solutions.*

Besides the axiomatic properties of *ME*, we are also interested in its set-theoretic relationships to other tournament solutions.

Theorem 3. *For all tournaments $T$, $ME(T) \subseteq BA(T)$.*

This also implies that the *irregularity* of *BA* [4, Theorem 7.1.3] extends to *ME*.

It is unknown whether the tournament equilibrium set is always contained in *ME* and whether *ME* is always contained in the minimal covering set.

### *Computational complexity.*

An important property of every tournament solution is whether it can be computed efficiently. By a reduction from 3SAT, we can show that this is not the case for *ME*.

Theorem 4. *Deciding whether an alternative in a tournament is contained in ME is NP-hard.*

Membership of the problem in NP seems rather unlikely. The best upper bound we know of is $\Sigma_3^p$.

## 4. CONCLUSION AND DISCUSSION

We have analyzed the axiomatic as well as computational properties of the tournament solution *ME*. Results were mixed. In conclusion, *ME*

(*i*) is not monotonic,
(*ii*) is not independent of unchosen alternatives,
(*iii*) satisfies $\widehat{\alpha}_{\supseteq}$ and idempotency,
(*iv*) does not satisfy $\widehat{\alpha}_{\subseteq}$ and $\widehat{\gamma}_{\supseteq}$ and is not stable,
(*v*) satisfies irregularity,
(*vi*) is contained in the Banks set,
(*vii*) is NP-hard to compute, and
(*viii*) satisfies composition-consistency [1].

It is worth pointing out that *ME*'s violation of monotonicity, stability, and independence of unchosen alternatives crucially depends on the existence of tournaments with more than one minimal extending set. Not only is the size of known tournaments of this type enormous (about $10^{136}$ alternatives) but, furthermore, these tournaments are very likely to be extremely rare. In effect, *ME does* satisfy these properties in all scenarios in which tournaments only admit a unique minimal extending set. Hence, it is fair to say that *ME* satisfies the considered properties for all practical purposes. This, in turn, may be interpreted as a criticism of the axiomatic method in general: For what does it mean if a tournament solution (or any other mathematical object) in principle violates some desirable properties, but no concrete example of a violation is known and will perhaps ever be known?

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