

# Generalized Matching Games for International Kidney Exchange

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## ABSTRACT

We introduce generalized matching games defined on a graph  $G = (V, E)$  with an edge weighting  $w$  and a partition  $V = V_1 \cup \dots \cup V_n$  of  $V$ . The player set is  $N = \{1, \dots, n\}$ , and player  $p \in N$  owns the vertices in  $V_p$ . The value  $v(S)$  of coalition  $S \subseteq N$  is the maximum weight of a matching in the subgraph of  $G$  induced by the vertices owned by players in  $S$ . If  $|V_p| = 1$  for every player  $p$  we obtain the classical matching game. We prove that checking core non-emptiness is polynomial-time solvable if  $|V_p| \leq 2$  for each  $p$  and co-NP-hard if  $|V_p| \leq 3$  for each  $p$ . We do so via pinpointing a relationship with  $b$ -matching games and also settle the complexity classification on testing core non-emptiness for  $b$ -matching games. We propose generalized matching games as a suitable model for international kidney exchange programs, where the vertices in  $V$  correspond to patient-donor pairs and each  $V_p$  represents one country. For this setting we prove a number of complexity results.

## KEYWORDS

kidney exchanges; matching game; core; computational complexity

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## 1 INTRODUCTION

The *assignment game* is a TU-game defined on a weighted bipartite graph, where the nodes are the agents and the value of a coalition is the maximum weight of a matching in the induced subgraph [33]. The core of any assignment game is always non-empty and can be computed efficiently [33]. The *matching game* is its generalization to non-bipartite graphs, where the core can be empty, but the problem of finding a core element (if it exists) is polynomial time solvable [9]. The *multiple partners assignment game* [35] and the *b-matching game* [10] are natural generalizations of the assignment and matching game, respectively, where the agents may be

involved in multiple pairs up to their capacities (i.e. we consider  $b$ -matchings). The core is again nonempty in the bipartite case [34]. For the non-bipartite case, deciding if a given allocation is in the core is co-NP-hard already with capacities  $b \leq 3$  and tractable for  $b \leq 2$  [10]. The complexity of deciding if the core of a  $b$ -matching game is nonempty was left open for  $b \leq 3$  in [10]. In Section 4 we solve this open problem by proving co-NP-hardness even for unit weights.

In Section 3 we introduce a second generalization of the assignment game, called *generalized matching game*, which is defined on a weighted (arbitrary) graph  $G$ , whose node set is partitioned into sets, and these sets form the agents of the game. The value of a coalition is again the maximum weight of a matching in the corresponding induced subgraph of  $G$ . We show a close relationship between generalized matching games and  $b$ -matching games regarding core non-emptiness. By combining this relationship with the results for  $b$ -matching games, we prove in Section 4 that testing core non-emptiness is co-NP-hard for generalized matching games in which each set has size at most 3, even for unit weights, and polynomial-time solvable if each set has size at most 2.

As a strong motivation for the generalized matching game we consider international kidney exchange schemes in Europe and multi-hospital exchange schemes in the US. In both cases the nodes represent patient-donor pairs, but the agents represent sets of countries in the first case and sets of hospitals in the second case. The matching edges correspond to pairwise kidney exchanges, where edge weights represent the quality of the transplants (or number of transplants in the unit weighted case). As “fair” target solutions we initially propose to take core solutions. This leads to the computational challenge of finding a maximum weight matching such that the utilities realized by the countries (or hospitals) are as close as possible to the target shares. In Section 5 we show the tractability of this problem for unit weights, but prove NP-hardness for various weighted scenarios. The deviation between the target and realized solutions are recorded as credits, which are taken into account in the subsequent matching runs; we assume that the matching runs take place in regular time intervals, e.g. in every three months (as usual in Europe). Below we discuss this application in more detail.

## 2 INTERNATIONAL KIDNEY EXCHANGE

For kidney failure, transplantation is currently the most effective treatment, but there is a shortage on deceased donors and waiting lists are long. A patient may have a willing donor, but a kidney transplant might not be possible due to blood- or tissue-type incompatibilities. However, patients and donors may be swapped after all patient-donor pairs are pooled together. A *kidney exchange program* (KEP) is a centralized program where the goal is to find an optimal kidney exchange scheme in some pool of patient-donor pairs.

One can model the above problem via a *compatibility graph*, which is a directed graph  $D = (V, A)$  with an arc weighting  $w$ . Each vertex in  $V$  represents a patient-donor pair, and there is an arc from patient-donor pair  $i$  to patient-donor pair  $j$  if the donor of pair  $i$  is compatible with the patient of pair  $j$ . The associated weight  $w_{ij}$  indicates the utility of the transplant. An *exchange cycle* is a directed cycle  $C$  in  $D$ . The *weight* of a cycle  $C$  is the sum of the weights of its arcs. An *exchange scheme*  $X$  is the union of pairwise vertex-disjoint exchange cycles of  $D$ . The weight of  $X$  is the sum of the weights of its cycles. The aim is to find an exchange scheme of maximum weight, subject to a fixed *exchange bound*  $\ell$ , which is an upper bound on the length of the exchange cycles that may be used. The reason for the latter is that kidneys are usually transplanted simultaneously and large exchange cycles may cause logistical difficulties.

Although KEPs are not legalized in some countries, national KEPs exist in many countries all over the world [20] including ten European countries [8]. For example, in the French and the Swedish KEPs the exchange bound is  $\ell = 2$  [3], whereas  $\ell = 3$  in the UK [24, 29] and  $\ell = 4$  in the Netherlands [14]. Setting  $\ell \geq 3$  changes the complexity of the problem from polynomial-time solvable, via solving a matching problem, to NP-hard [1]. In the latter case the problem is usually solved via integer programming techniques (see e.g. [1]). In fact, NP-hardness is not a major obstacle, as in many countries the size of the KEP pool (the set  $V$ ) is small. To find better solutions, one can merge KEP pools of different countries to obtain larger KEP pools. This leads to *international* KEPs, which are still in their initial stages. For instance, the pools of the Austrian and Czech KEPs have recently been joined [12]. ScandiTransplant will organise the international KEP of Sweden, Norway and Denmark [3]. Other examples include initial agreements between France and Switzerland, and between Portugal, Spain and Italy [7].

We model an international KEP by partitioning the vertex set  $V$  of a compatibility graph  $D = (V, A)$  into sets  $V_1, \dots, V_n$ , where  $n$  is the number of countries involved and  $V_p$  is the set of patient-donor pairs of country  $p$ . The objective is still to maximize *social welfare*, that is, to find an exchange scheme of  $D$  that has maximum weight subject to the given exchange bound  $\ell$ . We can compute such a scheme as before. However, apart from a number of ethical and legal issues which we will not discuss here, we now have an additional problem to solve. Namely, in order to ensure full participation, it is crucial that *proposed exchange schemes will be accepted by each of the participating countries*. This is a highly non-trivial issue.

*Example 1.* Let  $D$  be a compatibility graph with vertices  $i_1, i_2, j$  and arcs  $(i_1, i_2)$ ,  $(i_2, i_1)$ ,  $(i_2, j)$ ,  $(j, i_2)$  with weights  $1 - \epsilon$ ,  $1 - \epsilon$ ,  $1$ ,  $1$ , respectively, for some small  $\epsilon$ . Let  $V_1 = \{i_1, i_2\}$  and  $V_2 = \{j\}$ . The optimum solution is an exchange between  $i_2$  and  $j$  with weight 2. However, the in-house solution of  $V_1$  consisting of the exchange

between  $i_1$  and  $i_2$  (with weight  $2 - 2\epsilon$ ) is better for  $V_1$ , as then both patients in the pairs  $i_1$  and  $i_2$  receive a kidney, and with more or less the same chance of success, so,  $(i_1, i_2)$  is “easy-to-match” in-house.

Example 1 illustrates the problem of countries having an incentive to hide their easy-to-match pairs and only register their hard-to-match pairs to the international KEP. For instance, in the US large hospitals take up the role of “local KEPs” and conduct around 62% of the transplantations in-house and only 38% with the help of the three nationwide KEPs (UNOS, APD, NKR). This fragmentation is highly inefficient [2]. Proposed solutions use a matching mechanism ensuring that full hospital participation is individually rational [5, 6]. The tradeoff between optimality and strategy-proofness (with regard to reporting the full pools) has also been investigated in recent theoretical papers [4, 11, 37]. The same goal was behind the concept of a credit system, where hospitals are rewarded for disclosing their patient-donor pairs [23]. Indeed, among the three nationwide KEPs in the US, NKR is considered to be the most successful, partly due to their strong financial incentives for full participation and a credit system for patient-donor pair registration; each hospital is assigned a “Liquidity Score” based on the relative number of easy-to-match patients a hospital is bringing to the pool.

The kidney exchange collaborations of the European countries differ from the collaboration of US hospitals in many respects. In Europe the countries register their pools fully due to their strict national protocols. In the US this can only be achieved by giving incentives to the hospitals. In Europe the matching runs are typically conducted once every three months. In the US this is done more or less on-line on a daily basis. Both systems also have different health care practices, e.g. with respect to the use of desensitization.

The goal of the *international kidney exchange problem* is to offer kidney exchange schemes of maximum weight in the compatibility graph that are acceptable for each of the participating countries. Our goal is to provide a fundamental basis for this problem with a focus on the European setting. For this setting we also propose a credit system, but the above differences with the US setting explains why we will base our credit system on flexible game-theoretical fair shares rather than pre-defined scores for each type of patient-donor pair, as done in the US. We emphasize that our suggestion to use core allocations as initial target solutions for international cooperations is not to avoid *ad-hoc* blocking by coalitions of countries in the KEP but to guarantee fair, mutual *long-term* benefits for all parties. We describe our model in more detail in the next section, where we also discuss some related work that inspired our research [13, 26].

## 3 GAME-THEORETIC MODEL

A (*cooperative*) *game* is a pair  $(N, v)$ , where  $N$  is a set of  $n$  players and  $v : 2^N \rightarrow \mathbb{R}_+$  is a *value function* with  $v(\emptyset) = 0$ . If  $v(N) \geq v(S_1) + \dots + v(S_r)$  for every partition  $(S_1, \dots, S_r)$  of  $N$ , then the players may form the *grand coalition*  $N$ . Under this assumption, the central problem is then how to distribute  $v(N)$  amongst the players. An *allocation* is a vector  $x \in \mathbb{R}^N$  with  $x(N) = v(N)$ , where  $x(S) = \sum_{p \in S} x_p$  for  $S \subseteq N$ . The *core* of a game consists of all allocations  $x \in \mathbb{R}^N$  satisfying  $x(S) \geq v(S)$  for each  $S \subseteq N$ . Core allocations are highly desirable, as they offer no incentive for a subset  $S$  of players to leave  $N$  and form a coalition on their own. So core allocations ensure that the grand coalition  $N$  is *stable*. However, the

core may be empty, and the next problem may be computationally hard (assuming a “compact” description of the input).

**CORE NON-EMPTINESS**

*Instance:* A game  $(N, v)$ .

*Question:* Is the core of  $(N, v)$  nonempty?

We introduce the notion of a *generalized matching game*  $(N, v)$ , defined on an undirected graph  $G = (V, E)$  with a positive edge weighting  $w$  and partition  $(V_1, \dots, V_n)$  of  $V$ . We set  $N = \{1, \dots, n\}$ . For  $S \subseteq N$ , we let  $V(S) = \bigcup_{p \in S} V_p$ . The value  $v(S)$  of coalition  $S$  is the maximum weight of a matching in the subgraph of  $G$  induced by  $V(S)$ . If  $V_p = \{p\}$  for  $p = 1, \dots, n$ , then we obtain a *matching game* [9, 15, 17, 25, 27]. Hence, generalized matching games are matching games where one player may own more than one vertex. Such games are well suited to model the international kidney exchange problem. To explain this, we first assume that the exchange bound  $\ell = 2$ . The reason for this assumption is that  $\ell = 2$  is used in several countries and there is no universally agreed exchange bound. Moreover, for  $\ell = 2$ , we can compute a maximum weight exchange scheme in polynomial time. We modify a compatibility graph  $D = (V, A)$  into an undirected graph  $\bar{D} = (V, E)$  by adding an edge between two vertices  $i$  and  $j$  of  $V$  if and only if both  $(i, j)$  and  $(j, i)$  belong to  $A$ . We give each edge  $ij$  weight  $w(ij) = w_{ij} + w_{ji}$ . We obtain a maximum weight exchange scheme of  $(D, w)$  by computing a maximum weight matching in  $(\bar{D}, w)$ , which takes polynomial time [16]. We say that  $(\bar{D}, w)$  is the weighted graph that *underlies*  $(D, w)$ . For the international kidney exchange problem, a player  $p$  represents a country  $p$  with set of patient-donor pairs  $V_p$  and country size  $|V_p|$ . For  $S \subseteq N$ , the set  $V(S) = \bigcup_{p \in S} V_p$  is the union of all patient-donor pairs in the countries of  $S$ . We justify our model below.

The goal of an international KEP is to form the grand coalition  $N$  and to keep  $N$  stable. For a generalized matching game  $(N, v)$  it holds that  $v(N) \geq v(S_1) + \dots + v(S_r)$  for every partition  $(S_1, \dots, S_r)$  of  $N$ . However, as seen from Example 1, even 2-country coalitions may not be stable. Moreover, it is illegal to pay for kidneys and thus we cannot associate any monetary value with them. Hence, we cannot distribute the total value  $v(N)$  among the participating countries according to some allocation  $x$ . Just as proposed in [26], we overcome both obstacles by following the solution of the US and proposing a credit system. We do this, because optimal kidney exchange schemes are usually computed three or four times each year. Hence, we can level out discrepancies in single rounds so that, on average, social welfare *does* get allocated in a manner that encourages the participating countries to stay in the grand coalition.

For a certain round, let  $(N, v)$  be a generalized matching game defined by a compatibility graph  $(D, w)$  with country partition  $\mathcal{V}$ . Assume we are given a “fair” allocation  $y$  together with a *credit function*  $c : N \rightarrow \mathbb{R}$ , which satisfies  $\sum_{p \in N} c_p = 0$  where  $c_p$  is the *credit* that country  $p$  has received in the past. Then, for  $p = 1, \dots, n$ , we set  $x_p = y_p + c_p$ . Note that  $x$  is again an allocation, as  $y$  is an allocation and  $\sum_{p \in N} c_p = 0$ . Recall that we maximize social welfare and hence only consider the maximum weight matchings of  $(\bar{D}, w)$ . Let  $\mathcal{M}$  denote the set of all maximum weight matchings of  $(\bar{D}, w)$ . For  $p \in N$ , a *utility function*  $u_p$  gives for each  $M \in \mathcal{M}$ , a *utility*  $u_p(M)$ , which expresses the worth of  $M$  for  $p$ . The aim is to choose a maximum weight matching  $M \in \mathcal{M}$  with  $u_p(M)$  “as close

as possible” to  $x_p$  for each country  $p$ . Afterwards we compute a new credit function for the next round and repeat the process.

Note that we do not use allocations to distribute  $v(N)$ , but instead use them to find an acceptable sequence of maximum weight matchings  $M^0, M^1, M^2, \dots$  for all participating countries. To keep our model as general as possible, we did not specify the credit function  $c$ , utility function  $u$ , allocation  $y$  or norm  $\|\cdot\|$ . We give specific examples later and define the following problem.

**ALLOCATION APPROXIMATION**

*Instance:* A generalized matching game  $(N, v)$  defined by a compatibility graph  $(D, w)$  and partition  $\mathcal{V}$ ; an allocation  $x$ ; and a constant  $\delta$ .

*Question:* Does  $(\bar{D}, w)$  have a maximum weight matching  $M$  such that  $\|x_p - u_p(M)\| \leq \delta$  for  $p = 1, \dots, n$ ?

This problem is trivial for graphs with a unique maximum weight matching, which is highly likely when weights  $w_{ij}$  take many different values at random [30]. However, in our context, we mainly consider compatibility graphs with only a *small* number of different weights. The reason is that to overcome certain blood and antigen incompatibilities, patients can undergo one or more desensitization treatments to match with their willing donor. After full desensitization the chance on a successful kidney transplant is almost the same as in the case of full compatibility. Allowing desensitization results in compatibility graphs with weights either 1 (when no desensitization was needed) or  $1 - \epsilon$  (after applying desensitization). As  $\epsilon$  is small, it is sometimes even assumed that  $w \equiv 1$ .

*All features of our model are present in the forthcoming international KEP between Sweden, Norway, and Denmark, where  $\ell = 2$ , desensitization is possible, and the size of the solution is the first priority [3].*

**Related Work.** Carvalho et al. [13] also modelled international KEPs using game theory. They mainly considered the situation with two countries,  $\ell = 2$ , no credit system, and matching runs over two stages. In the first stage each country decides which in-house exchanges they conduct and in the second stage a maximum matching is selected for the patient-donor pairs registered for the international exchange. Klimentova et al. [26] considered international KEPs with a credit system. The differences with our model are as follows: 1) they allow  $\ell \geq 3$ , whereas we set  $\ell = 2$ ; 2) they use a particular individually rational solution concept for computing fair allocations based on marginal contributions, whereas we suggest the core of the corresponding generalized matching game; and 3) they consider only the size of the solutions, whereas we also investigate the weighted case, where the scores represent the utilities of the transplants. They also performed simulations using integer programming techniques for investigating the long-term effects of their compensation policy.

Gourvès, Monnot and Pascua [21] considered a variant of generalized matching games where organizations own a number of vertices in a market situation. Their goal differs from ours and is to find an individually rational maximum weight matching (which gives each organization  $p$  at least the value that it can obtain on its own). They also proved complexity results in this setting for several parameters, such as the number of organizations, number of weights and maximum degree.

## 4 CORE NON-EMPTINESS

Here we show our results for CORE NON-EMPTINESS for  $b$ -matching and generalized matching games. For a vertex capacity function  $b$ , a  $b$ -matching in an undirected graph  $G = (V, E)$  is a subset  $M \subseteq E$  such that each  $i \in V$  is incident to at most  $b_i$  edges in  $M$ . A  $b$ -matching game is a game  $(N, v)$  on an undirected graph  $G = (N, E)$  with edge weighting  $w$ , such that for  $S \subseteq N$ ,  $v(S)$  is the maximum weight of a  $b$ -matching in the subgraph of  $G$  induced by  $S$ . A matching game is a 1-matching game. It is well known that CORE NON-EMPTINESS is polynomial-time solvable for matching games; see [9] for an  $O(nm + n^2 \log n)$ -time algorithm. In [10] it was shown that deciding if an allocation belongs to the core of a  $b$ -matching game is polynomial-time solvable if  $b \leq 2$  and co-NP-complete if  $b \equiv 3$ . The first result implies that CORE NON-EMPTINESS is polynomial-time solvable for  $b$ -matching games with  $b \leq 2$  [10]. However, the case where  $b \not\leq 2$  was left open. We prove it is co-NP-hard even if  $w \equiv 1$  and  $b_i \leq 3$  for every  $i \in N$ . By pinpointing a relationship with generalized matching games, we also prove that CORE NON-EMPTINESS problem is co-NP-hard for generalized matching games even when  $w \equiv 1$  and country sizes  $\leq 3$ . As such, we first show the following reduction.

**THEOREM 4.1.** *The CORE NON-EMPTINESS problem for generalized matching games with country size  $\leq c$  reduces to the CORE NON-EMPTINESS problem for  $b$ -matching games with capacities  $b \leq c$ .*

**PROOF.** We assume  $c \geq 2$  as for  $c = 1$  both problems are identical. Let  $(N, v)$  be a generalized matching game defined by a graph  $G = (V, E)$  with edge weights  $w$  and partition  $\mathcal{V} = (V_1, \dots, V_n)$  of the vertex set. We construct a corresponding  $b$ -matching game  $(\bar{N}, \bar{v})$ , defined by a graph  $\bar{G} = (\bar{N}, \bar{E})$  (where  $\bar{N} \supseteq V$  and  $\bar{E} \supseteq E$ ), edge weights  $w$  and node capacities  $b$  as follows. For each  $V_i$ , we add a new root node  $r_i$  that is adjacent to all nodes in  $V_i$  and no other nodes in  $G$ . Thus in total we add  $n$  new nodes and  $|V|$  new edges. Every new edge gets weight  $2W$  where  $W > v(N)$ . Let  $R$  be the set of root nodes. This completes our description of  $\bar{G} = (\bar{N}, \bar{E})$  on vertex set  $\bar{N} = V \cup R$ . All nodes in  $V$  get capacity  $b = 2$  and each node  $r_i \in R$  gets capacity  $|V_i|$ . Let  $(\bar{N}, \bar{v})$  be the corresponding  $b$ -matching game. We claim that  $(N, v)$  has non-empty core if and only if  $(\bar{N}, \bar{v})$  has so.

“ $\Rightarrow$ ” Suppose  $x \in \text{core}(N, v)$ . For  $i = 1, \dots, n$ , we let  $\bar{x} := \frac{x_i}{|V_i|} + W$  on  $V_i$  and  $\bar{x}_{r_i} := |V_i|W$ . We claim that  $\bar{x} \in \text{core}(\bar{N}, \bar{v})$ . Indeed,  $\bar{x}(\bar{N}) = \bar{v}(\bar{N}) = v(N) + |V|2W$  by definition. To check the core constraints, consider  $\bar{S} \subseteq \bar{N}$ . Let  $S := \{i \mid \bar{S} \cap V_i \neq \emptyset\}$ . A maximum weight  $b$ -matching in  $\bar{G}[\bar{S}]$  is obtained by matching each root node  $r_i \in \bar{S}$  to all its neighbors in  $\bar{S}$  and matching the nodes in  $\bar{S} \cap V$  to each other in the best possible way. Thus  $\bar{v}(\bar{S}) \leq v(S) + \sum_{i:r_i \in \bar{S}} |\bar{S} \cap V_i|2W$ , while  $\bar{x}(\bar{S}) = \sum_{i \in S} \left( \frac{|\bar{S} \cap V_i|}{|V_i|} x_i + |\bar{S} \cap V_i|W \right) + \sum_{i:r_i \in \bar{S}} |V_i|W$ . Comparing the two values, we find that the core constraint  $\bar{x}(\bar{S}) \geq \bar{v}(\bar{S})$  holds unless  $\bar{S} = \bigcup_{i \in S} V_i \cup \{r_i\}$ . In the latter case, however,  $\bar{v}(\bar{S}) = v(S) + \sum_{i \in S} |V_i|2W$  and  $\bar{x}(\bar{S}) = x(S) + \sum_{i \in S} |V_i|2W$ , so that the core constraint follows from  $x(S) \geq v(S)$ .

“ $\Leftarrow$ ” Assume  $\text{core}(N, v) = \emptyset$ . By the Bondareva-Shapley Theorem, there are coalitions  $S_q \subseteq N$  and  $\lambda_q \geq 0$  such that  $\sum_q \lambda_q S_q = N$  and  $\sum_q \lambda_q v(S_q) > v(N)$  (here, for convenience, we identify coalitions

$S_q$  and  $N$  with their corresponding incidence vectors in  $\mathbb{R}^n$ ). Define corresponding coalitions  $\bar{S}_q := \bigcup \{V_i \cup \{r_i\} \mid i \in S_q\}$  in  $(\bar{N}, \bar{v})$ . A maximum weight  $b$ -matching in  $\bar{G}[\bar{S}_q]$  is obtained by matching each root  $r_i \in \bar{S}_q$  to all nodes in  $V_i$  and matching  $\bar{S}_q \cap V$  in an optimal way. Thus  $\bar{v}(\bar{S}_q) = v(S_q) + |V_q|2W$ . Hence, again writing coalitions as incidence vectors,  $\sum_q \lambda_q \bar{S}_q = \sum_q \lambda_q (\sum_{i \in S_q} V_i + \{r_i\}) = \sum_i (\sum_{q:i \in S_q} \lambda_q) (V_i + \{r_i\}) = \sum_i (V_i + \{r_i\}) = \bar{N}$  and, similarly,  $\sum_q \lambda_q \bar{v}(\bar{S}_q) = \sum_q \lambda_q v(\bigcup_{i \in S_q} V_i \cup \{r_i\}) = \sum_q \lambda_q v(S_q) + |V|2W > v(N) + |V|2W = \bar{v}(\bar{N})$ , showing that also  $\text{core}(\bar{N}, \bar{v}) = \emptyset$ .  $\square$

As CORE NON-EMPTINESS is polynomial-time solvable for  $b$ -matching games with  $b \leq 2$  [10], we obtain the following result.

**COROLLARY 4.2.** *CORE NON-EMPTINESS is polynomial time solvable for generalized matching games with country size  $\leq 2$ .*

Contrary to above, our next reduction reduces instances with uniform weights  $w = 1$  to instances with uniform weights.

**THEOREM 4.3.** *The CORE NON-EMPTINESS problem for  $b$ -matching games with  $b \leq c$  reduces to the CORE NON-EMPTINESS problem for generalized matching games with country sizes  $\leq c$ . The transformation can be done so that uniform weight instances of  $b$ -matching are transformed to uniform weight instances of generalized matching.*

**PROOF.** Let  $(N, v)$  be a  $b$ -matching game defined by  $G = (V, E)$ , edge weights  $w$  and node capacities  $b$ . We construct a weighted graph  $\bar{G} = (\bar{V}, \bar{E})$  with partition  $\mathcal{V}$  of its vertex set such that the corresponding generalized matching game has a non-empty core if and only if  $\text{core}(N, v)$  is non-empty. To this end we apply a classical construction of Tutte [38] that is generally used to reduce  $b$ -matching to standard matching problems. This works as follows. Each node  $i \in V$  of capacity  $b_i$  gets replaced by  $b_i$  copies  $i^{(s)}$ ,  $s = 1, \dots, b_i$ . Secondly, each edge  $ij \in E$  gets replaced by a tree  $T_{ij}$  connecting the copies of  $i$  to the copies of  $j$ . The tree consists of a central edge with endpoints  $i_j$  and  $j_i$ . Node  $i_j$  is adjacent to all copies of  $i$  and, similarly,  $j_i$  is adjacent to all copies of  $j$  (see also Figure 1). All edges in  $T_{ij}$  get weight  $w_{ij}$ . Denote the resulting graph by  $\bar{G} = (\bar{V}, \bar{E})$ . The idea is that any  $b$ -matching  $M$  in  $G$  can be represented by a corresponding matching  $\bar{M} \subseteq \bar{E}$  in  $\bar{G}$  as follows. If  $e = ij \in M$ , then we match  $i_j$  to some copy of  $i$  in  $\bar{G}$  and, similarly,  $j_i$  to some copy of  $j$ . (Note that, by definition, enough copies of  $i$  resp.  $j$  are available.) If  $e = ij \notin M$ , then we match  $i_j$  and  $j_i$  to each other in  $\bar{G}$ . The resulting matching  $\bar{M}$  in  $\bar{G}$  then has size  $|E| + |M|$  and weight  $w(E) + w(M)$ . We refer to  $\bar{M}$  as a *transform* of  $M$ . (Different transforms differ by the choice of copies of node  $i$  that are “matched” to  $j$ .) The partition of  $\bar{V}$  is the obvious one with blocks  $V_i$  consisting of all copies of  $i$  and 2-node blocks  $E_{ij} = \{i_j, j_i\}$ . This completes the description of the generalized matching game  $(\bar{N}, \bar{v})$ . Note that the players in  $\bar{N}$  are in 1-1 correspondence with  $V \cup E = N \cup E$ , so we sometimes also identify them with  $V \cup E$ . We claim that  $\text{core}(N, v)$  is non-empty if and only if  $\text{core}(\bar{N}, \bar{v})$  is non-empty.

“ $\Rightarrow$ ” Let  $x \in \text{core}(N, v)$ , assume  $M$  is a maximum weight  $b$ -matching in  $G$  (so  $v(N) = w(M)$ ) and define an allocation  $\bar{x}$  on  $\bar{N}$  by setting  $\bar{x}(V_i) := x_i$  for  $i \in N$  and  $\bar{x}(E_{ij}) = w_{ij}$ . Then  $\bar{x} \in \text{core}(\bar{N}, \bar{v})$ . Indeed, first observe that a maximum weight matching in  $\bar{G}$  is a transform  $\bar{M}$  of  $M$ , so  $\bar{v}(\bar{N}) = w(\bar{M}) = v(N) + w(E)$ . Thus  $\bar{x}(\bar{N}) = \bar{v}(\bar{N})$

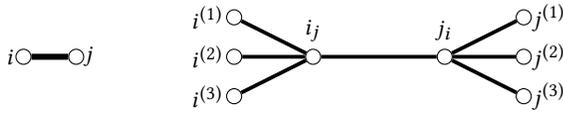


Figure 1: Tutte's gadget replacing an edge  $e = ij$ .

indeed. To check the core constraints, consider a coalition  $S \subseteq \bar{N}$ . Let  $\bar{S} \subseteq \bar{V}$  be the union of all blocks in  $S$ . Then  $\bar{v}(S)$  is the weight of a maximum weight matching in  $\bar{G}[\bar{S}]$ . The latter is obtained as a transform of a maximum weight  $b$ -matching  $M$  in  $G[\bar{S} \cap V]$ . So  $\bar{v}(S) = w(M) + w(E[\bar{S}])$  and  $\bar{x}(S) = \sum_{i,j: E_{ij} \subseteq \bar{S}} w_{ij} + \sum_{i: V_i \subseteq \bar{S}} x_i = w(E[\bar{S}]) + x(S \cap V) \geq \bar{v}(S)$  because  $x(S \cap V) \geq w(M)$  by assumption. “ $\Leftarrow$ ” Assume  $\text{core}(N, v) = \emptyset$ . By the Bondareva-Shapley Theorem, there are  $\lambda_q \geq 0$  and  $S_q \subseteq V$  such that  $\sum \lambda_q S_q = N$  and  $\sum \lambda_q v(S_q) > v(N)$ . Let  $\bar{S}_q := \bigcup \{V_i \mid i \in S_q\} \cup \bigcup \{E_{ij} \mid i, j \in S_q\}$ . Then (the incidence vector of)  $\sum \lambda_q \bar{S}_q$  equals 1 on  $N = V$  and is at most 1 on  $\bar{N} \setminus N = E$ . By setting  $\lambda_{ij} = 1 - \sum_{q: E_{ij} \subseteq \bar{S}_q} \lambda_q$ , we construct a non-negative combination  $\bar{N} = \sum_q \lambda_q S_q + \sum_{ij} \lambda_{ij} E_{ij}$ . To show that  $\bar{v}(\bar{N}) < \sum_q \lambda_q \bar{v}(\bar{S}_q) + \sum_{ij} \lambda_{ij} \bar{v}(E_{ij})$ , let  $M_q$  be a maximum weight  $b$ -matching in  $G[\bar{S}_q]$ . Then  $\bar{M}_q$  has weight  $w(M_q) + w(E[\bar{S}_q])$  in  $\bar{G}[\bar{S}_q]$ . So  $\bar{v}(\bar{S}_q) \geq v(S_q) + w(E[\bar{S}_q])$ . Hence  $\sum_q \lambda_q \bar{v}(\bar{S}_q) + \sum_{ij} \lambda_{ij} \bar{v}(E_{ij}) \geq \sum_q \lambda_q (v(S_q) + w(E[\bar{S}_q])) + \sum_{ij} \lambda_{ij} w_{ij} > v(N) + \sum_e (\sum_{q: e \in E(S_q)} \lambda_q) w_e + \sum_{ij} \lambda_{ij} w_{ij} = v(N) + w(E) = \bar{v}(\bar{N})$ .  $\square$

We identify a  $b$ -matching  $M$  in a graph  $G$  with the subgraph of  $G$  induced by  $M$  (subgraph of  $G$  consisting of all edges in  $M$  and vertices covered by  $M$ ). We speak about (connected) components of  $M$ . For instance, for  $b = 1$ , every edge  $e \in M$  is a component.

LEMMA 4.4. *Let  $(N, v)$  be a  $b$ -matching game on weighted graph  $(G, w)$  with a nonempty core. Let  $x$  be a core allocation of  $(N, v)$  and  $M$  be a maximum weight matching of  $G$ . Then, for every component  $C$  of  $M$ , it holds that  $x(C) = w(C)$ .*

THEOREM 4.5. *CORE NON-EMPTINESS is co-NP-hard for  $b$ -matching games with  $b \leq 3$ , even if  $w \equiv 1$ . The same holds for (uniform weight) generalized matching games with country size  $\leq 3$ .*

PROOF. Due to Theorem 4.3 it suffices to prove the first statement. The proof is by reduction from the 3-REGULAR SUBGRAPH problem, which is to decide if a given graph has a 3-regular subgraph (a graph is 3-regular if every vertex has degree 3). This problem is NP-complete even for bipartite graphs [36]. Actually, we use a slight variant that might be called the NEARLY 3-REGULAR SUBGRAPH problem: given a (non-bipartite) graph, decide if it has a subgraph with all nodes of degree 3 except for one node of degree 2. This is NP-complete as well: given an instance of 3-REGULAR SUBGRAPH, i.e., a bipartite graph  $(U \cup V, E)$ , construct the non-bipartite graph  $G$  consisting of  $|E|$  disjoint copies of  $(U \cup V, E)$  where in the copy corresponding to  $e \in E$  the edge  $e$  is subdivided by a new node, say  $v_e$ . Then  $(U \cup V, E)$  has a 3-regular subgraph if and only if  $G$  has a nearly 3-regular subgraph. Indeed, if there is a 3-regular subgraph in  $(U \cup V, E)$  that contains the edge  $e$ , there will be an almost 3-regular subgraph in  $G$  whose degree 2 node is  $v_e$ . Conversely, if there is an almost 3-regular subgraph in  $G$ , it must contain a node

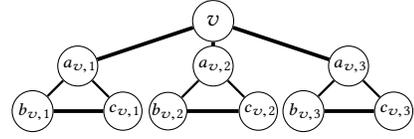


Figure 2: Attached triangles  $(a_v, b_v, c_v)$  and  $r$  omitted.

$v_e$  for some  $e$ , because otherwise the subgraph would be bipartite, but an almost 3-regular graph cannot be bipartite.

We reduce from NEARLY 3-REGULAR SUBGRAPH for non-bipartite graphs. Given an instance  $G = (V, E)$  of the latter, we construct a graph  $\bar{G}$  with vertex capacities  $b_i \leq 3$  and edge weights  $w = 1$  such that  $G$  has a nearly 3-regular subgraph if and only if the weighted  $b$ -matching game on  $(\bar{G}, w)$  has an empty core. We construct  $\bar{G}$  as follows. To every vertex  $v$  of  $G$  we attach three edges  $va_{v,1}$ ,  $va_{v,2}$  and  $va_{v,3}$ . Each of the three new vertices  $a_{v,j}$ , for  $j = 1, 2, 3$ , is part of a triangle with vertices  $a_{v,j}$ ,  $b_{v,j}$ ,  $c_{v,j}$ . Vertex capacities are  $b_v = 3$  for all “original” vertices  $v \in V$  and  $b_w = 2$  on all new “triangle” vertices  $w$ . Finally, for each  $v \in V$  there is a vertex  $a_v$  of capacity  $b = 3$  that is adjacent to all three vertices  $a_{v,j}$ , for  $j = 1, 2, 3$ . Similarly, there are vertices  $b_v$  and  $c_v$ , adjacent to  $b_{v,j}$ , for  $j = 1, 2, 3$  and  $c_{v,j}$ , for  $j = 1, 2, 3$ , resp. Finally, we add a root node  $r$  that is adjacent to all  $v \in V$  and none of the other (new) nodes. The root  $r$  has capacity  $b_r = 1$ . This completes the description of  $\bar{G} = (\bar{V}, \bar{E})$  with corresponding vertex capacities  $b$  and edge weights  $w = 1$ . See also Figure 2.

We next describe a maximum weight matching (as indicated in Figure 2) in  $\bar{G}$ . Let  $M$  consist of all edges  $va_{v,j}$  plus all edges of the form  $b_{v,j}c_{v,j}$  plus all edges incident to  $a_v$ ,  $b_v$  and  $c_v$ . Thus  $M$  saturates all nodes except  $r$ , so  $M$  is a maximum (weight) matching.

First suppose  $G$  contains no nearly 3-regular subgraph. We claim that in this case  $x \equiv \frac{3}{2}$  on the vertices in  $V$ ,  $x \equiv 1$  on the vertices of each triangle,  $x \equiv \frac{3}{2}$  on the “connector” vertices  $a_v, b_v, c_v$  ( $v \in V$ ) and  $x_r = 0$  yields a core allocation. Obviously we have  $x(\bar{V}) = w(M) = |M|$ . To show that  $x$  satisfies the core constraints, suppose to the contrary that there exists a blocking coalition, i.e., a vertex set  $S \subseteq \bar{V}$  with corresponding maximum weight matching  $M_S$  in the subgraph induced by  $S$  such that  $x(S) < |M_S|$ . Assume furthermore that  $S$  is a (w.r.t. set inclusion) minimal blocking coalition. Since  $x_i$  equals half the capacity of each vertex except  $r$ , this can only happen if  $S$  contains  $r$  and  $M_S$  saturates all vertices in  $S$ . So, in particular  $M_S$  contains some edge  $rv_0$ ,  $v_0 \in V$ . As  $M_S$  saturates all nodes in  $S$ ,  $v_0$  must be matched by  $M_S$  to two more nodes (other than  $r$ ). Assume first that all  $v \in S \cap V \setminus \{v_0\}$  are either matched “down” to  $a_{v,1}, a_{v,2}, a_{v,3}$  by three matching edges in  $M_S$  or matched “up” by three matching edges  $e \in M_S \cap E$ . If  $v \in V \cap S$  is matched down to its three triangles, then the component of  $M$  containing these three edges joining  $v$  to its triangles is paid exactly its value (all vertices in the component are saturated and each vertex gets exactly half of its capacity except  $v$ ). Removing the component of  $M$  containing  $v$  from  $S$  thus results in a smaller blocking set  $S' \subset S$ , contradicting the minimality of  $S$ . Thus, all vertices  $v \in S \cap V$ ,  $v \neq v_0$  must be matched “up”. If also  $v_0$  is matched “up” by two edges in  $M_S \cap E$ , then  $(S \cap V, M_S)$  is a nearly 3-regular subgraph of  $G$ , a contradiction. So we are left to deal with the case where there exists a vertex

$v \in S \cap V$  that is, say, matched down by some edge  $e = va_{v,1} \in M_S$  but, say,  $e' = va_{v,3} \notin M_S$ . We distinguish the following cases:

**Case 1.**  $a_v, b_v, c_v \in S$ . Since all these are saturated by  $M_S$ , we have all  $a_{v,j}, b_{v,j}, c_{v,j} \in S$ . Thus  $a_{v,3}, b_{v,3}, c_{v,3} \in S$  and each of these is already matched to  $a_v, b_v, c_v$ , resp. Since  $va_{v,3} \notin M_S$ , at most two of  $a_{v,3}, b_{v,3}, c_{v,3}$  can be saturated by  $M_S$ , a contradiction.

**Case 2.**  $a_v, b_v \in S, c_v \notin S$ . Again we find that  $a_{v,1}, a_{v,2}, a_{v,3} \in S$  and  $b_{v,1}, b_{v,2}, b_{v,3} \in S$ . Moreover, each of these is already matched by some edge in  $M_S$  to  $a_v$  or  $b_v$ . In addition,  $a_{v,1}$  is matched to  $v$ , so  $a_{v,1}$  is “already” saturated. Hence, in order to saturate also  $b_{v,1}$ ,  $M_S$  must contain  $b_{v,1}c_{v,1}$ . Hence,  $c_{v,1} \in S$  and  $M_S$  cannot saturate it (as  $c_v \notin S$ ), a contradiction.

**Case 3.**  $a_v \in S, b_v, c_v \notin S$ . Here we conclude  $a_{v,1}, a_{v,2}, a_{v,3} \in S$ . Since  $va_{v,3} \notin M_S$ ,  $a_{v,3}$  can only be saturated if, say,  $a_{v,3}b_{v,3} \in M_S$  and hence  $b_{v,3} \in S$ . The latter can only be saturated by  $b_{v,3}c_{v,3} \in M_S$ . Hence,  $c_{v,3} \in S$  and this cannot be saturated (since  $a_{v,3}b_{v,3} \in M_S$  would imply that  $a_{v,3}$  is already saturated from edges inside its triangle, so  $a_v$  cannot be saturated any more), a contradiction.

**Case 4.**  $a_v \notin S$ . Since  $a_{v,1}$  is in  $S$ , it must be saturated, and as  $a_v \notin S$ , either  $a_{v,1}b_{v,1}$  or  $a_{v,1}c_{v,1} \in M_S$ . By symmetry, suppose that  $a_{v,1}b_{v,1} \in M_S$ . Then  $b_{v,1}$  is in  $S$  and must be saturated, so either  $b_{v,1}b_v \in M_S$  (and  $c_{v,1}$  must be uncovered by  $M_S$ ) or  $b_{v,1}c_{v,1} \in M_S$ . In the first case  $b_v \in S$  and  $c_v \notin S$ , in the second case  $b_v \notin S$  and  $c_v \in S$  (as  $c_{v,1}$  can only be saturated by  $c_{v,1}c_v$ ). In both cases we get a contradiction when considering the third triangle, as follows. If  $b_v \in S$  and  $c_v \notin S$ , then we have  $b_vb_{v,3} \in M_S$ . Thus  $b_{v,3}$  is in  $S$  and must be saturated, i.e., matched to either  $a_{v,3}$  or  $c_{v,3}$ . In the first case  $a_{v,3}$  must be matched to  $c_{v,3}$  and the latter remains unsaturated, a contradiction. In the second case,  $c_{v,3}$  must be saturated by matching it to  $a_{v,3}$  and then again, the latter must remain unsaturated. The case  $c_v \in S$  and  $b_v \notin S$  is similar. From  $c_v \in S$  we conclude that  $c_vb_{v,3} \in M_S$ . Thus  $c_{v,3} \in S$  and this must be matched to either  $a_{v,3}$  or  $b_{v,3}$ . In the first case,  $a_{v,3}$  must be matched to  $b_{v,3}$  (as  $a_v$  is not available) and  $b_{v,3}$  remains unsaturated. In the second case  $b_{v,3}$  must be matched to  $a_{v,3}$  and again, the latter remains unsaturated, a contradiction.

Now suppose the  $b$ -matching game on  $\bar{G}$  has a core allocation  $x$ . Fix any  $v \in V$  and let  $S_{ab} := \{a_v, b_v, a_{v,j}, b_{v,j} \mid j = 1, 2, 3\}$ . As  $S_{ab}$  allows a saturating matching  $M_{ab}$  of size  $|M_{ab}| = 9$ , we find that  $x(S_{ab}) \geq 9$ . Similarly,  $x(S_{bc}) \geq 9$  and  $x(S_{ac}) \geq 9$  for  $S_{bc}$  and  $S_{ac}$  defined analogously. Adding all three inequalities and dividing by 2 yields  $x(S) \geq 27/2$  for  $S := S_{ab} \cup S_{bc} \cup S_{ac}$ . The set  $S \cup \{v\}$  is covered exactly by two components of the maximum weight matching  $M$  in  $\bar{G}$ . Hence, by Lemma 4.4 we obtain  $x(S \cup \{v\}) = 15$  and  $x_r = 0$ , so  $x_v \leq \frac{3}{2}$ . As this holds for all  $v \in V$ , any nearly 3-regular subgraph  $G' = (S, F)$  of  $G$  with distinguished node  $v_0$  of degree 2 would define a blocking coalition  $S \cup \{r\}$ . Indeed, the edge set  $F \cup \{rv_0\}$  matches each node in  $S$  up to its capacity, while  $x$  assigns only half this value to each node in  $S$  and zero to  $r$ , implying  $x(S) < v(S)$ , contrary to our assumption that  $x$  is in the core. So there can be no nearly 3-regular subgraph.  $\square$

## 5 ALLOCATION APPROXIMATION

Recall that to keep the ALLOCATION APPROXIMATION problem as general as possible, we did not specify the credit function  $c$ , utility function  $u$ , allocation  $x$  and distance norm  $\|\cdot\|$ . We note that  $c$  is

irrelevant for our problem and that  $x$  is part of the input (although we argued to let  $x$  be a core allocation). Hence, we only need to define the utility function  $u$  and norm  $\|\cdot\|$ . As norm we choose the classical norm  $|a - b|$  for two numbers  $a, b$ . As to the utilities  $u_p(M)$ , there are two natural options.

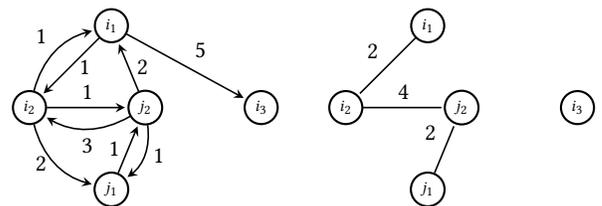
**Cardinalities.** We may define  $u_p(M)$  as the total number of incoming kidneys for country  $p$  by  $M \in \mathcal{M}$ . That is, let  $u_p(M) = s_p(M)$  for  $p = 1, \dots, n$ , where  $s_p(M)$  is the size of the set  $M_D(p) = \{(i, j) \in A \mid ij \in M, j \in V_p\}$ , or equivalently,  $s_p(M) = |\{j \in V_p \mid \exists i \in V : ij \in M\}|$ . See Figure 3 for an example. This is a natural utility function due to its simplicity and because in practice the weights  $w_{ij}$  are sparsely spread (see Section 3). Using  $s_p$  also has a computational advantage. Namely, we prove that for  $u_p = s_p$ , ALLOCATION APPROXIMATION is polynomial-time solvable. For example, given an allocation  $x$  and constant  $\delta$ , we can find in polynomial time some  $M \in \mathcal{M}$  (if it exists) such that  $s_p(M) \in [x_p - \delta, x_p + \delta]$  for  $p = 1, \dots, n$ .

**Weights.** We may define  $u_p(M)$  as the total weight of the incoming kidneys for  $p$ . That is, let  $u_p(M) = t_p(M)$  for  $p = 1, \dots, n$ , where  $t_p(M) = \sum_{i,j:ij \in M, j \in V_p} w_{ij}$  (see also Figure 3). If  $w \equiv 1$  on  $D$ , then  $t_p = s_p$  and we can solve ALLOCATION APPROXIMATION in polynomial time, see above. If  $n = 1$ , then the problem is trivially polynomial-time solvable, as  $t_1(M)$  is the same for every  $M \in \mathcal{M}$ . If all country sizes are 1, we obtain a matching game, and we will also prove polynomial-time solvability. However, if the number of different weights is 2 in both  $(D, w)$  and  $(\bar{D}, w)$  or if  $n = 2$ , then we prove NP-hardness. We also prove NP-hardness if country sizes are  $\leq 2$ , but only if we assume some compact description of the input. The case where  $w \equiv 1$  on  $\bar{D}$  (but  $w \not\equiv 1$  in  $D$ ) turns out to be polynomially equivalent with EXACT PERFECT MATCHING [31], a well-known problem whose complexity is yet unknown.

We prove the following result for  $u_p = s_p$ ; a similar construction was used by Plesnik [32] to solve a constrained matching problem.

**THEOREM 5.1.** *Given a generalized matching game  $(N, v)$  on a weighted graph  $(G, w)$ , and closed intervals  $I_1, \dots, I_n$ , it is possible in polynomial time to decide if there exists a matching  $M \in \mathcal{M}$  with  $s_p(M) \in I_p$  for  $p = 1, \dots, n$ , and to find such a matching (if exists).*

**PROOF.** Let  $w^*$  be the maximum weight of a matching in  $G$ . Let  $I_p = [a_p, b_p]$ , where  $b_p \leq |V_p|$ . We extend  $G = (V, E)$  to a graph  $\bar{G}$  as follows. For  $p = 1, \dots, n$ , we add a set  $B_p$  of  $|V_p| - b_p$  new



**Figure 3: A compatibility graph  $(D, w)$  and its undirected graph  $(\bar{D}, w)$ . Let  $M_1 = \{i_2j_2\}$  and  $M_2 = \{i_1i_2, j_1j_2\}$ . Then  $w(M_1) = w(M_2) = 4$ , and  $\mathcal{M} = \{M_1, M_2\}$ . Let  $V_1 = \{i_1, i_2, i_3\}$  and  $V_2 = \{j_1, j_2\}$ . Then  $s_1(M_1) = s_2(M_1) = 1$ , and  $t_1(M_1) = 3$  and  $t_2(M_1) = 1$ , whereas  $s_1(M_2) = s_2(M_2) = t_1(M_2) = t_2(M_2) = 2$ .**

vertices, each of them joined to all vertices of  $V_p$  by edges of weight  $\bar{w}_e = 0$ . We also introduce a set  $A_p$  of  $b_p - a_p$  new vertices that are completely joined to all vertices of  $V_p$  by edges of weight  $\bar{w}_e = 0$ . In addition, all vertices in  $\bigcup_p A_p$  are joined to each other by edges of weight  $\bar{w}_e = 0$ . The original edges  $e \in E$  in  $G$  keep their (original) weights, i.e.,  $\bar{w}_e = w_e$ . In case the total number of vertices is odd, we add an additional vertex  $\bar{v}$  and join it by zero weight edges to all vertices of  $\bigcup_p A_p$ . This completes the description of  $(\bar{G}, \bar{w})$ . Let  $\bar{w}^*$  denote the maximum weight of a perfect matching in  $\bar{G}$ , which we can compute in polynomial time [16, 28]. Hence, it suffices to show there is a matching  $M \in \mathcal{M}$  with  $s_p(M) \in [a_p, b_p]$  for  $p = 1, \dots, n$  if and only if  $\bar{w}^* = w^*$ .

“ $\Rightarrow$ ” Suppose there is a matching  $M \in \mathcal{M}$  with  $s_p(M) \in [a_p, b_p]$  for  $p = 1, \dots, n$ . As  $M \in \mathcal{M}$ , we have  $w(M) = w^*$ . As  $s_p \leq b_p$ , we can match all vertices of  $B_p$  to  $V_p$  by all zero weight edges. Finally, since  $s_p \geq a_p$ , we can match all (at most  $b_p - a_p$ ) remaining vertices in  $V_p$  from  $A_p$ . Thus, eventually, all vertices of  $V_p$  will get matched. In case there are vertices in  $\bigcup_p A_p$  that are not yet matched, we match these to each other and, in case their number is odd, to the extra vertex  $\bar{v}$ . This yields a perfect matching in  $\bar{G}$  of weight  $\bar{w}^*$ .

“ $\Leftarrow$ ” Suppose  $\bar{w}^* = w^*$ . Let  $\bar{M}$  be a corresponding perfect matching in  $\bar{G}$  of weight  $\bar{w}^*$ . Let  $M := \bar{M} \cap E$  denote the corresponding matching in  $G$ . As  $\bar{M}$  matches all vertices of  $B_p$  into  $V_p$ , we know that  $M$  leaves at least  $|V_p| - b_p$  vertices unmatched. Hence,  $s_p(M) \leq b_p$  as required. Similarly, since all vertices of  $V_p$  are matched by  $\bar{M}$  and at most  $|V_p| - b_p + b_p - a_p = |V_p| - a_p$  vertices in  $V_p$  can be matched to  $B_p \cup A_p$ , we find that  $M$  matches at least  $a_p$  vertices in  $V_p$ , so  $s_p(M) \geq a_p$ , as required.  $\square$

**COROLLARY 5.2.** For  $u_p = s_p$ , ALLOCATION APPROXIMATION is polynomial-time solvable.

We now consider the case where  $u_p = t_p$  and recall that for  $u_p = t_p$ , ALLOCATION APPROXIMATION is polynomial-time solvable if  $n = 1$  or  $w \equiv 1$  on  $D$ . We show the following result (proof omitted).

**THEOREM 5.3.** For  $u_p = t_p$ , ALLOCATION APPROXIMATION is polynomial-time solvable for matching games, or equivalently, if all country sizes are 1.

In what follows below, some instances will make use of reductions from the NP-complete problem PARTITION [19], which is to decide if there is a set  $I \subseteq \{1, \dots, k\}$  with  $a(I) = \frac{1}{2} \sum_{i=1}^k a_i$  for some given tuple of  $k$  integers  $a_1, \dots, a_k$ .

**THEOREM 5.4.** For  $u_p = t_p$ , ALLOCATION APPROXIMATION is NP-complete even if  $n = 2$ .

**PROOF.** We show the statement even for  $\delta = 0$ . We reduce from PARTITION. From an instance  $(a_1, \dots, a_k)$  of PARTITION we construct a generalized matching game  $(N, v)$  with  $n = 2$ . We define countries  $V_1 = \{v_1, \dots, v_k, v'_1, \dots, v'_k\}$  and  $V_2 = \{v''_1, \dots, v''_k\}$ . For  $i = 1, \dots, k$  we have arcs  $(v_i, v'_i)$ ,  $(v'_i, v_i)$ ,  $(v_i, v''_i)$  and  $(v''_i, v_i)$ , each with weight  $a_i$ . Any maximum weight matching  $M$  matches each  $v_i$  with either  $v'_i$  or  $v''_i$ . Matching  $v_i$  with  $v'_i$  adds  $2a_i$  to country  $V_1$ 's utility (and 0 to the utility of  $V_2$ ), while matching  $v_i$  with  $v''_i$  adds  $a_i$  to both the utility of  $V_1$  and  $V_2$ . Note that  $v(N) = 2 \sum_j a_j$ . Let  $x$  be the allocation with  $x_1 = \frac{3}{2} \sum_j a_j$  and  $x_2 = \frac{1}{2} \sum_j a_j$ . Then

there exists a matching  $M \in \mathcal{M}$  with  $t_1(M) = x_1$  and  $t_2(M) = x_2$  if and only if  $(a_1, \dots, a_k)$  is a yes-instance of PARTITION.  $\square$

As in the setting of international KEPs sparsely weighted games are relevant, in the remainder of our paper we consider such cases.

**THEOREM 5.5.** For  $u_p = t_p$ , ALLOCATION APPROXIMATION is NP-complete even if the number of weights in the computability graph and its underlying graph is 2.

**PROOF.** We show the statement even for  $\delta = 0$ . We reduce from 3-PARTITION, which is to decide if we can partition a set of  $3k$  positive integers  $a_1, \dots, a_{3k}$ , with  $\sum_{p=1}^{3k} a_p = kc$  for some integer  $c$ , into  $k$  sets that each sum up to  $c$ . This problem is strongly NP-complete (so NP-complete even when encoded in unary) even if  $\frac{1}{4}c < a_i < \frac{1}{2}c$ , ensuring that each set in a solution has size exactly 3 [19].

From an instance  $(a_1, \dots, a_{3k})$  with  $\frac{1}{4}c < a_i < \frac{1}{2}c$  we construct a generalized matching game  $(N, v)$  on a compatibility graph  $(D, A)$  as follows. We start with  $3k$  sources. For  $p = 1, \dots, k$  let  $S_p := \{s_p, s'_p, s''_p\}$  and  $S := \bigcup_p S_p$ . Add a set of  $3k$  sinks  $T := \{z_1, \dots, z_{3k}\}$ . Join all sources to all sinks by  $(3k)^2$  pairwise internally vertex disjoint paths: from each  $s_p(s'_p, s''_p)$  there is a path  $P_{pq}(P'_{pq}, P''_{pq})$  to each  $z_q$  of length  $2a_q - 1$ . Any two consecutive vertices on the path are joined by two opposite arcs of equal weight. The weights on each path alternate between  $L$  and  $L + 1$ , starting and ending with  $L + 1$ , where  $L \gg 0$  is sufficiently large, say,  $L > kc$ . For  $p = 1, \dots, k$ , let  $V_p = \bigcup_q (P_{pq} \cup P'_{pq} \cup P''_{pq}) \setminus T$  and let  $V_{k+1} = T$ .

As  $L \gg 0$ , every maximum weight matching  $M$  in the underlying graph  $\bar{D} = (V, E)$  is perfect. More precisely,  $M$  looks as follows. For  $p = 1, \dots, k$  there are three paths  $P_{pq}$  from  $s_p$  to  $z_q$ ,  $P'_{pq}$  from  $s'_p$  to  $z_q$ , and  $P''_{pq}$  from  $s''_p$  to  $z_q$  that are completely matched in the sense that  $M \cap P_{pq}$  is a perfect matching of  $P_{pq}$  (and similarly for  $P'_{pq}$  and  $P''_{pq}$ ), contributing a gain of  $(2(a_q + a'_q + a''_q) - 3)(L + 1)$  to  $u_p(M)$ . Furthermore, there are  $(3k - 3)$  paths from  $s_p$  to the remaining  $3k - 3$  sinks in  $T \setminus \{z_q, z'_q, z''_q\}$  that start and end with a non-matching edge (and are otherwise  $M$ -alternating). These paths (emanating from  $s_p$ ) contribute a total of  $2L(\sum_{r \notin \{q, q', q''\}} (a_r - 1)) = 2L(\sum_r a_r - (a_q + a'_q + a''_q) - (3k - 3))$  to  $u_p(M)$ . So  $t_p(M) = 2(a_q + a'_q + a''_q) + 2L(\sum a_r) - 6L(k - 1)$  for  $p = 1, \dots, k$ . Let  $x$  be the allocation with  $x_p = 2c + 2L(\sum a_r) - 6L(k - 1)$  for  $p = 1, \dots, k$  and  $x_{k+1} = 3k((3k - 1)L + L + 1)$ . Then there is a matching  $M \in \mathcal{M}$  with  $t_p(M) = x_p$  for  $p = 1, \dots, k + 1$  if and only if  $(a_1, \dots, a_{3k})$  is a yes-instance of 3-PARTITION. As 3-PARTITION is strongly NP-complete,  $a_1, \dots, a_{3k}$  can be represented in unary. Thus, the size of the instance of 3-PARTITION is  $kc$ . Hence,  $(D, w)$  has polynomial size.  $\square$

Note that the number of countries in Theorem 5.5 can be arbitrarily large. By a “compact description” of a game defined on a graph we mean a logarithmic description of the graph (if possible). For example, a cycle of length  $k$  can be described by its length, which results in input size  $O(\log k)$  rather than  $k$ .

**THEOREM 5.6.** For  $u_p = t_p$ , ALLOCATION APPROXIMATION is NP-complete even for compact generalized matching games with three different weights and country sizes  $\leq 2$ .

**PROOF.** We show the statement even for  $\delta = 0$ . We reduce again from the NP-complete PARTITION problem [19]. From an

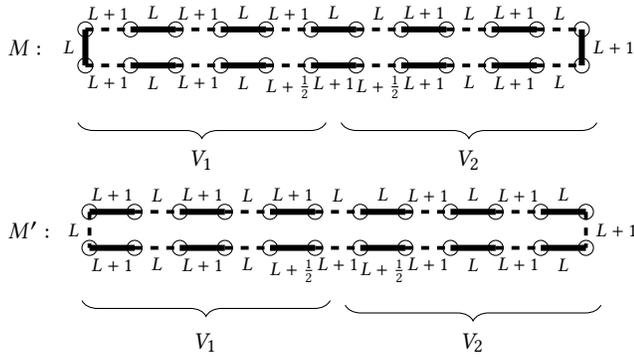


Figure 4:  $C = C_i$  for  $a_i = 5$  with edges  $e$  and  $\bar{e}$  in the middle.

instance  $(a_1, \dots, a_k)$  of PARTITION we construct a compact generalized matching game  $(N, v)$  with number of weights 3 and country sizes  $\leq 2$ . We assume that  $k$  is even (otherwise add  $a_{k+1} = 0$ ), that the size of  $I$  is  $|I| = k/2$  (otherwise add a large number to each  $a_i$ ) and that every  $a_i$  is odd (otherwise replace every  $a_i$  by  $2a_i + 1$ ).

Let  $C = C_i$  be an even cycle of length  $4a_i + 4$ . Let  $e$  and  $\bar{e}$  be two opposite edges. Assign weights  $w_e = L$  and  $w_{\bar{e}} = L + 1$  to these edges, where  $L > 0$  is large, say,  $L = \sum a_i$ . Weights  $w_e$  and  $w_{\bar{e}}$  are assumed to be split equally to their corresponding two opposite arcs. Removing  $e$  and  $\bar{e}$  splits  $C$  into two paths  $P_1$  and  $P_2$  of length  $2a_i + 1$  each. The edge weights on these two paths alternate between  $L$  and  $L + 1$  except for their last edge, which has weight  $L + \frac{1}{2}$ . More precisely,  $P_1$  starts with an edge (say, incident to  $e$ ) of weight  $L + 1$  and continues alternating between edges of weight  $L + 1$  and  $L$  until its last edge (incident to  $\bar{e}$ ) gets weight  $L + \frac{1}{2}$  (instead of  $L + 1$ ). Similarly,  $P_2$  starts with an edge of weight  $L$ , incident to  $e$ , and alternates between weights  $L + 1$  and  $L$  until the last edge gets weight  $L + \frac{1}{2}$  (instead of  $L$ ). See Figure 4 for the case where  $a_i = 5$ .

We let  $U_1$  and  $U_2$  denote the vertex sets of  $P_1$  and  $P_2$ , respectively. For  $L$  suitably large,  $C$  has exactly two maximum weight matchings, namely its two complementary perfect matchings  $M$  and  $M'$ , where  $M$  is the perfect matching that matches both  $e$  and  $\bar{e}$  and  $M'$  is the complement of  $M$ . We compute:  $t_1(M) = \frac{1}{2}L + \frac{1}{2}(L + 1) + a_iL = L(a_i + 1) + \frac{1}{2}$ ,  $t_2(M) = \frac{1}{2}L + \frac{1}{2}(L + 1) + a_i(L + 1) = L(a_i + 1) + \frac{1}{2} + a_i$ ,  $t_1(M') = L(a_i + 1) + \frac{1}{2} + a_i$ , and  $t_2(M') = L(a_i + 1) + \frac{1}{2}$ .

Recall that we have  $k$  such components  $C_i$ , each with two complementary maximum weight (perfect) matchings. So in the graph  $G$  consisting of these  $k$  components  $C_i$  we have  $2^k$  maximum weight matchings, obtained by picking one of the two complementary  $M$  and  $\bar{M}$  in each  $C_i$ . Let  $V_1$  be the union of all the  $U_1$ s in each  $C_i$  and  $V_2$  be the union of all the  $U_2$ s. Consider the allocation  $x$  with  $x_1 = x_2 = L(\sum a_i + 1) + \frac{1}{2} \sum a_i + k/2$  and assume these can be realized by a suitable maximum matching. Let  $I \subseteq \{1, \dots, k\}$  be the set of indices  $i$  such that the matching picks  $M$  in  $C_i$ . With respect to this matching,  $V_1$  has utility  $L \sum (a_i + 1) + k/2 + \sum_I a_i$ . Such a matching exists if and only if  $(a_1, \dots, a_k)$  is a yes-instance of PARTITION. This completes the reduction. Each component  $C_i$  of the graph we construct has a description of length  $O(\log(ka_{max}))$ , where  $a_{max}$  denotes the maximum  $a_i$ ; note that  $L$  is bounded by  $\log(ka_{max})$  and

the length of  $C_i$  is bounded by  $a_i$ . Hence, allowing compact descriptions, the weighted graph we constructed has size  $O(k \log(ka_{max}))$ , which is polynomial in the size of  $(a_1, \dots, a_k)$ .  $\square$

We now consider the case where  $n = 2$  and  $w \equiv 1$  on  $(\bar{D}, w)$  but the computability graph  $(D, w)$  itself has two different weights. We do not solve this case, but link it to EXACT PERFECT MATCHING introduced in [31]. This problem has as input an undirected graph  $G$  whose edge set is partitioned into a set  $R$  of *red* edges and a set  $B$  of *blue* edges. The question is whether  $G$  has a perfect matching with exactly  $k$  red edges for some given integer  $k$ . The complexity status of EXACT PERFECT MATCHING is a longstanding open problem, and so far only partial results were shown (see, for example, [22]).

Let  $D = (V, A)$  be a compatibility graph  $D = (V, A)$ , in which all 2-cycles on vertices  $i, j$  have weights  $w_{ij} = \frac{1}{3}$  and  $w_{ji} = \frac{2}{3}$ . Note that  $w \equiv 1$  in the underlying weighted graph  $(\bar{D}, w)$ . In fact the exact values of  $w_{ij}$  and  $w_{ji} = 1 - w_{ij}$  do not matter, as long as they differ from  $\frac{1}{2}$  (if  $w \equiv \frac{1}{2}$  on  $D$ , then  $t_p(M) = s_p(M)$  and we can apply Corollary 5.2). Let  $(V_1, V_2)$  be the country partition, such that  $i \in V_1, j \in V_2$  implies that  $w_{ij} = \frac{1}{3}$  and  $w_{ji} = \frac{2}{3}$ . Note that edges inside  $V_1$  and  $V_2$  are also allowed. Moreover, we assume that  $\bar{D}$  has a perfect matching. As  $w \equiv 1$  in  $\bar{D}$ , the set  $\mathcal{M}$  of maximum weight matchings of  $\bar{D}$  consists of all perfect matchings. We call the generalized matching game  $(N, v)$  defined on such a compatibility graph  $(D, w)$  and  $(V_1, V_2)$  *perfect*. We show the following result (proof omitted).

**THEOREM 5.7.** EXACT PERFECT MATCHING and ALLOCATION APPROXIMATION on perfect generalized matching games  $(N, v)$  are polynomially equivalent.

## 6 CONCLUSIONS

Just as for other cooperative games (such as, flow games [18]), we generalized matching games by allowing a player to own multiple vertices. We showed that generalized matching games are equivalent to  $b$ -matching games with respect to CORE NON-EMPTYNESS and proved two complexity dichotomies. For the case with only 2-way kidney exchanges, we used these games to model a credit system in international Kidney Exchange Programs, introducing a credit system for compensating unhappy countries in future rounds. This led to the ALLOCATION APPROXIMATION problem for computing exchange schemes as close as possible to some given allocation. If the total number of incoming kidneys is the utility function, we gave a polynomial-time algorithm. If instead their total weight is taken as the utility function, we proved NP-hardness. For the latter case the main open problem is to determine the complexity in case of small country sizes; we could only show NP-hardness for compact encodings.

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