Many-to-Many Stable Matchings with Ties, Master Preference Lists, and Matroid Constraints

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ABSTRACT

In this paper, we consider a matroid generalization of the hospitals/residents problem with ties. Especially, we focus on the situation in which we are given a master list and the preference list of each hospital over residents is derived from this master list. In this setting, Kamiyama proved that if hospitals have matroid constraints and each resident is assigned to at most one hospital, then we can solve the super-stable matching problem and the strongly stable matching problem in polynomial time. In this paper, we generalize these results to the many-to-many setting. More specifically, we consider the setting where each resident can be assigned to multiple hospitals, and the set of hospitals that this resident is assigned to must form an independent set of a matroid. In this paper, we prove that the super-stable matching problem and the strongly stable matching problem in this setting can be solved in polynomial time.

KEYWORDS

Stable matching; Matroid; Tie

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1 INTRODUCTION

The two-sided matching market model proposed by Gale and Shapley [7] is one of the most fundamental mathematical models for real-world assignment problems. In this model, each agent has a preference list over potential partners. Gale and Shapley [7] proved that if there does not exist a tie in the preference lists (i.e., the preference lists are strict), then there always exists a stable matching and we can find a stable matching in polynomial time.

However, if there exist ties in the preference lists, then the situation dramatically changes (see, e.g., [12] and [21, Chapter 3] for a survey of stable matchings with ties). For the stable matching problem with ties, three stability concepts were proposed by Irving [8]. The first concept is called weak stability. This stability concept guarantees that there does not exist an unmatched pair $\{a, b\}$ such that a (resp., b) prefers b (resp., a) to the current partner. Irving [8] proved that there always exists a weakly stable matching and we can find a weakly stable matching in polynomial time by slightly modifying the algorithm of Gale and Shapley [7]. The second concept is called strong stability. This stability concept guarantees that there does not exist an unmatched pair $\{a, b\}$ such that (i) a prefers b to the current partner, and (ii) b prefers a to the current partner, or is indifferent between a and the current partner. The last concept is called super-stability. This stability concept guarantees that there does not exist an unmatched pair $\{a, b\}$ such that a (resp., b) prefers b (resp., a) to the current partner, or is indifferent between b (resp., a) and the current partner.

One of the most notable differences between the last two concepts and the stability concept in the stable matching problem with strict preferences is that there does not necessarily exist a stable matching [8]. From the algorithmic viewpoint, it is important to consider the problem of checking the existence of matchings satisfying these stability conditions. In the one-to-one setting, Irving [8] proposed polynomial-time algorithms for finding a super-stable matching and a strongly stable matching (see also [20]). In the many-to-one setting, Irving, Manlove, and Scott [9] proposed a polynomial-time algorithm for finding a super-stable matching, and Irving, Manlove, and Scott [10] proposed a polynomial-time algorithm for finding a strongly stable matching. Kavitha, Mehlhorn, Michail, and Paluch [17] proposed faster algorithms for strong stability. In the many-to-many setting, Scott [26] considered super-stability, and Malhotra [19] and Chen and Ghosh [2] considered strong stability. Olaosebikanand and Manlove [23] considered super-stability in the student-project allocation problem with ties.

In this paper, we focus on the situation in which we are given a master list and the preference list of each hospital over residents is derived from this master list. Master lists are motivated by real-world applications (see [11]). In the one-to-one setting, Irving, Manlove, and Scott [11] gave simple polynomial-time algorithms for finding a super-stable matching and a strongly stable matching. O'Malley [24] gave polynomial-time algorithms for finding a super-stable matching and a strongly stable matching in the many-to-one setting. Furthermore, Kamiyama [14] gave polynomial-time algorithms for finding a super-stable matching and a strongly stable matching in the many-to-one setting with matroid constraints. Matroids can represent not only capacity constraints but also more complex constraints including hierarchical capacity constraints. Thus, matroid generalizations are important from not only the theoretical viewpoint but also the practical viewpoint. Matroid generalizations of several matching problems have been extensively studied (see, e.g., [4–6, 13–16, 18, 22, 27]).

In this paper, we consider the problem of finding a many-to-many super-stable matching and a many-to-many strongly stable matching with master preference lists and matroid constraints. Our results generalize the results of Kamiyama [14]. More specifically, in

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the setting of [14], each resident is assigned to at most one hospital. On the other hand, in our setting, each resident can be assigned to multiple hospitals, and the set of hospitals that this resident is assigned to must form an independent set of a matroid. Notice that the extension from the many-to-one case to the many-to-many case is generally a non-trivial task (see, e.g., [1, 2]).

2 PRELIMINARIES

For each set X and each element x, we define $X + x := X \cup \{x\}$ and $X - x := X \setminus \{x\}$, respectively. For each positive integer n, we define $[n] := \{1, 2, \ldots, n\}$. Define $[0] := \emptyset$.

A pair $\mathbf{M} = (U, \mathcal{I})$ of a finite set U and a non-empty family $\mathcal{I}$ of subsets of U is called a *matroid* if for every pair of subsets I, J of U, the following conditions are satisfied.

- **(I1)** If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.
- **(I2)** If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists an element u in $J \setminus I$ such that $I + u \in \mathcal{I}$.

A subset of U belonging to $\mathcal{I}$ is called an *independent set of* $\mathbf{M}$.

In this paper, we are given a finite simple (not necessarily complete) bipartite graph $G = (V, E)$ such that its vertex set V is partitioned into disjoint subsets R and H, and each edge in E connects a vertex in R and a vertex in H. We call a vertex in R (resp., H) a *resident* (resp., *hospital*). For each resident r in R and each hospital h in H, if there exists an edge in E connecting r and h, then we denote by (r, h) this edge. For each vertex v in V and each subset F of E, we denote by $F(v)$ the set of edges in F that are incident to v. For each resident r in R, we are given a matroid $\mathbf{P}_r = (E(r), \mathcal{F}_r)$ such that for every edge e in $E(r)$, $\{e\} \in \mathcal{F}_r$. Furthermore, we are given a matroid $\mathbf{Q} = (E, \mathcal{G})$ such that for every edge e in E, $\{e\} \in \mathcal{G}$. We assume that we can decide whether each subset of E is an independent set of the above matroids in time bounded by a polynomial in the input size of G. (That is, we consider the oracle model.)

For each resident r in R, we are given a complete and transitive binary relation $\succsim_r$ on $E(r)$. Furthermore, we are given a complete and transitive binary relation $\succsim_H$ on R. For each resident r in R and each pair of edges e, f in $E(r)$ such that $e \succsim_r f$ and $f \not\succsim_r e$ (resp., $e \succsim_r f$ and $f \succsim_r e$), we write $e \succ_r f$ (resp., $e \sim_r f$). For each pair of residents r, s in R such that $r \succsim_H s$ and $s \not\succsim_H r$ (resp., $r \succsim_H s$ and $s \succsim_H r$), we write $r \succ_H s$ (resp., $r \sim_H s$).

A subset M of E is called a *matching* in G if the following conditions are satisfied.

- **(M1)** $M(r) \in \mathcal{F}_r$ for every resident r in R.
- **(M2)** $M \in \mathcal{G}$.

For each matching M in G and each edge $e = (r, h)$ in $E \setminus M$, we say that r *weakly prefers* (resp., *strongly prefers*) e on M if at least one of the following conditions is satisfied.

- **(R1)** $M(r) + e \in \mathcal{F}_r$.
- **(R2)** There exists an edge f in $M(r)$ such that $M(r) + e - f \in \mathcal{F}_r$ and $e \succsim_r f$ (resp., $e \succ_r f$).

For each matching M in G and each edge $e = (r, h)$ in $E \setminus M$, we say that H *weakly prefers* (resp., *strongly prefers*) e on M if at least one of the following conditions is satisfied.

- **(H1)** $M + e \in \mathcal{G}$.
- **(H2)** There exists an edge (s, p) in M such that $M + e - (s, p) \in \mathcal{G}$ and $r \succsim_H s$ (resp., $r \succ_H s$).

A matching M in G is said to be *super-stable* if there does not exist an edge (r, h) in $E \setminus M$ such that r and H weakly prefer (r, h) on M. A matching M in G is said to be *strongly stable* if there does not exist an edge (r, h) in $E \setminus M$ such that r and H weakly prefer (r, h) on M, and at least one of r and H strongly prefers (r, h) on M. The goal of the super-stable (resp., strongly stable) matching problem is to decide whether there exists a super-stable (resp., strongly stable) matching in G, and find a super-stable (resp., strongly stable) matching if a super-stable (resp., strongly stable) matching exists.

2.1 Notation

We denote by $R_1, R_2, \ldots, R_n$ the partition of R satisfying the following conditions.

- For every integer i in $[n]$ and every pair of residents r, s in R_i, $r \sim_H s$.
- For every pair of integers i_1, i_2 in $[n]$ such that $i_1 < i_2$ and every pair of residents r in R_{i_1} and s in R_{i_2}, $r \succ_H s$.

Then for each integer i in $[n]$, we define $R[i] := \bigcup_{x=1}^{i} R_x$.

For each resident r in R, we denote by $E_r^1, E_r^2, \ldots, E_r^{m_r}$ the partition of $E(r)$ satisfying the following conditions.

- For every integer i in $[m_r]$ and every pair of edges e, f in E_r^i, $e \sim_r f$.
- For every pair of integers i_1, i_2 in $[m_r]$ such that $i_1 < i_2$ and every pair of edges e in $E_r^{i_1}$ and f in $E_r^{i_2}$, $e \succ_r f$.

Then for each resident r in R and each integer i in $[m_r]$, we define $E_r[i] := \bigcup_{x=1}^{i} E_r^x$. For each resident r in R, we define $E_r[0] := \emptyset$.

2.2 Example

Here we give an example of our model. Assume that we are given a positive integer u_r for each resident r in R, and we are given a positive integer u_h for each hospital h in H. Then for each resident r in R, we define $\mathcal{F}_r$ as the family of subsets F of $E(r)$ such that $|F| \leq u_r$. Furthermore, we define $\mathcal{G}$ as the family of subsets F of E such that $|F(h)| \leq q_h$ for every hospital h in H. Then a subset M of E is a matching in G if the following conditions are satisfied.

- $|M(r)| \leq u_r$ for every resident r in R.
- $|M(h)| \leq u_h$ for every hospital h in H.

Assume that we are given a matching M in G and an edge $e = (r, h)$ in $E \setminus M$. Then r weakly (resp., strongly) prefers e on M if at least one of the following conditions is satisfied.

- $|M(r)| < u_r$.
- $e \succsim_r f$ (resp., $e \succ_r f$) for some edge f in $M(r)$.

Furthermore, H weakly (resp., strongly) prefers e on M if at least one of the following conditions is satisfied.

- $|M(h)| < u_h$.
- $r \succsim_H s$ (resp., $r \succ_H s$) for some edge (s, h) in $M(h)$.

Thus, our problem in this setting can be regarded as a many-to-many generalization of the hospitals/residents problem with ties and master lists.

3 MATROIDS

Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$. A subset C of U is called a *circuit* of $\mathbf{M}$ if C is not an independent set of $\mathbf{M}$, but every proper subset of C is an independent set of $\mathbf{M}$. The following property of circuits is known.

Lemma 3.1 (See, e.g., [25, Lemma 1.1.3]). *Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$. Then for every pair of distinct circuits C_1, C_2 of $\mathbf{M}$ such that $C_1 \cap C_2 \neq \emptyset$ and every element u in $C_1 \cap C_2$, there exists a circuit C of $\mathbf{M}$ such that $C \subseteq (C_1 \cup C_2) \setminus \{u\}$.*

Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$ and an independent set I of $\mathbf{M}$. It is not difficult to see that for every element u in $U \setminus I$ such that $I + u \notin \mathcal{I}$, $I + u$ contains a circuit of $\mathbf{M}$ as a subset, and (I1) implies that u belongs to this circuit. Furthermore, Lemma 3.1 implies that such a circuit is uniquely determined. We call this circuit the *fundamental circuit* of u with respect to I and $\mathbf{M}$, and we denote by $\mathrm{C}_{\mathbf{M}}(u, I)$ this circuit. It is well known (see, e.g., [25, p.20, Exercise 5]) that for every element u in $U \setminus I$ such that $I + u \notin \mathcal{I}$, $\mathrm{C}_{\mathbf{M}}(u, I)$ is the set of elements w in $I + u$ such that $I + u - w \in \mathcal{I}$. For each element u in $U \setminus I$ such that $I + u \notin \mathcal{I}$, we define $\mathrm{D}_{\mathbf{M}}(u, I) := \mathrm{C}_{\mathbf{M}}(u, I) - u$.

For each matching M in G and each edge $e = (r, h)$ in $E \setminus M$ such that $M(r) + e \notin \mathcal{F}_r$, (R2) can be restated as follows.

(R2) $e \succsim_r f$ (resp., $e \succ_r f$) for some edge f in $\mathrm{D}_{\mathbf{P_r}}(e, M(r))$.

For each matching M in G and each edge $e = (r, h)$ in $E \setminus M$ such that $M + e \notin \mathcal{G}$, (H2) can be restated as follows.

(H2) $r \succsim_H s$ (resp., $r \succ_H s$) for some edge (s, p) in $\mathrm{D}_{\mathbf{Q}}(e, M)$.

We can easily prove the following lemma by Lemma 3.1.

Lemma 3.2. *Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$, independent sets I, J of $\mathbf{M}$ such that $I \subseteq J$, and an element u in $U \setminus J$ such that $I + u \notin \mathcal{I}$. Then $J + u \notin \mathcal{I}$ and $\mathrm{C}_{\mathbf{M}}(u, I) = \mathrm{C}_{\mathbf{M}}(u, J)$.*

Lemma 3.3 (See, e.g., [14, Lemma 2]). *Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$, circuits $C, C_1, C_2, \ldots, C_x$ of $\mathbf{M}$, and distinct elements $u_1, u_2, \ldots, u_x$ in U satisfying the following conditions.*

- *$u_i \in C \cap C_i$ for every integer i in $[x]$.*
- *$u_{i_1} \notin C_{i_2}$ for any pair of distinct integers i_1, i_2 in $[x]$.*
- *$C \setminus (C_1 \cup C_2 \cup \cdots \cup C_x) \neq \emptyset$.*

Then there exists a circuit C' of $\mathbf{M}$ such that C' is a subset of $(C \cup C_1 \cup C_2 \cup \cdots \cup C_x) \setminus \{u_1, u_2, \ldots, u_x\}$.

Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$. Then a maximal independent set of $\mathbf{M}$ is called a *base* of $\mathbf{M}$. The condition (I2) implies that all bases of $\mathbf{M}$ have the same size. For each subset X of U, we define $\mathcal{I}|X$ as the family of subsets I of X such that $I \in \mathcal{I}$, and we define $\mathbf{M}|X := (X, \mathcal{I}|X)$. It is known [25, p.20] that for every subset X of U, $\mathbf{M}|X$ is a matroid. For each subset X of U, we define $\mathbf{r_M}(X)$ as the size of a base of $\mathbf{M}|X$. Define $\mathbf{r}(\mathbf{M}) := \mathbf{r_M}(U)$. Furthermore, for each pair of disjoint subsets X, J of U, we define $\mathrm{p}(J; X)$ as $\mathbf{r_M}(J \cup X) - \mathbf{r_M}(X)$. For each subset X of U, we define $\mathcal{I}/X$ as the family of subsets I of $U \setminus X$ such that $\mathrm{p}(I; X) = |I|$, and we define $\mathbf{M}/X := (U \setminus X, \mathcal{I}/X)$. It is known [25, Proposition 3.1.6] that for every subset X of U, $\mathbf{M}/X$ is a matroid.

Lemma 3.4 (See, e.g., [25, Proposition 3.1.25]). *Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$. Then for every pair of disjoint subsets X, Y of U, $(\mathbf{M}/X)|Y = (\mathbf{M}|(X \cup Y))/X$.*

Lemma 3.5 (See, e.g., [15, Lemma 1]). *Assume that we are given a matroid $\mathbf{M} = (U, \mathcal{I})$, a subset X of U, and a base B of $\mathbf{M}|X$. Then for every subset I of $U \setminus X$, I is an independent set (resp., a base) of $\mathbf{M}/X$ if and only if $I \cup B$ is an independent set (resp., a base) of $\mathbf{M}$.*

Assume that we are given k matroids $\mathbf{M}_1 = (U_1, \mathcal{I}_1), \ldots, \mathbf{M}_k = (U_k, \mathcal{I}_k)$ such that $U_1, U_2, \ldots, U_k$ are pairwise disjoint. Define

$$\bigoplus_{i=1}^{k} \mathcal{I}_i := \left\{ X \subseteq \bigcup_{i=1}^{k} U_i \;\middle|\; X \cap U_i \in \mathcal{I}_i \text{ for every integer } i \text{ in } [k] \right\}.$$

$$\bigoplus_{i=1}^{k} \mathbf{M}_i := \left(\bigcup_{i=1}^{k} U_i,\ \bigoplus_{i=1}^{k} \mathcal{I}_i \right).$$

It is not difficult to see that $\bigoplus_{i=1}^{k} \mathbf{M}_i$ is a matroid.

Assume that we are given two matroids $\mathbf{M}_1 = (U, \mathcal{I}_1)$ and $\mathbf{M}_2 = (U, \mathcal{I}_2)$. A subset I of U is called a *common independent set* of $\mathbf{M}_1$ and $\mathbf{M}_2$ if $I \in \mathcal{I}_1 \cap \mathcal{I}_2$. It is well known (see, e.g., [3]) that we can find a maximum-size common independent set of $\mathbf{M}_1$ and $\mathbf{M}_2$ in time bounded by a polynomial in $|S|$ and EO, where EO is the time required to decide whether X is an independent set of $\mathbf{M}_i$ for every subset X of U and every integer i in $\{1, 2\}$.

4 SUPER-STABLE MATCHINGS

In this section, we propose an algorithm for the super-stable matching problem (see Algorithm 1). This algorithm is based on the algorithm of [14] for the super-stable matching problem in the many-to-one setting with matroid constraints. For proving that Algorithm 1 is a polynomial-time algorithm, it is sufficient to prove that we can decide whether each subset of E is an independent set of the matroids in Algorithm 1 in time bounded by a polynomial in the input size of G. We can easily prove this by Lemma 3.5 as follows. At Line 7 of Algorithm 1, to check whether $\{e\}$ is an independent set of $\mathbf{Z}_r/D_r[i-1]$ for each edge e in $E_r^i \cap A_{t-1}$, it suffices to find a base B of $\mathbf{Z}_r|D_r[i-1]$ and check whether $\{e\} \cup B$ is an independent set of $\mathbf{Z}_r$. If the time complexity of the independence oracle for the given matroids is EO, then it is not difficult to see that we can implement Algorithm 1 in $O(n|E|\mathsf{EO})$ time, where we assume that $\mathsf{EO} = \Omega(|E|)$ and $\max\{|R|, |H|\} \leq |E|$.

What remains is to prove the correctness of Algorithm 1. In the rest of this section, we assume that Algorithm 1 halts when $t = k$.

Lemma 4.1. *If Algorithm 1 outputs M_n, then M_n is a super-stable matching in G.*

Proof. Define $M := M_n$. For every resident r in R, since $M(r) = T_r$, Lines 12 and 13 of Algorithm 1 imply that $M(r) \in \mathcal{F}_r$. Furthermore, Lines 16 to 19 imply that $M \in \mathcal{G}$. Thus, M is a matching in G. What remains is to prove that M is super-stable. Let $e = (r, h)$ be an edge in $E \setminus M$. Notice that $e \notin T_r$. Assume that $r \in R_z$.

We first assume that $e \notin A_{z-1}$. Then there exists an integer ℓ in $[z-1]$ such that $e \in L_\ell$. Thus, $M_\ell + e \notin \mathcal{G}$ and $s \succ_H r$ for every edge (s, p) in $\mathrm{D}_{\mathbf{Q}}(e, M_\ell)$. Furthermore, since $M_\ell \subseteq M$, Lemma 3.2 implies that $M + e \notin \mathcal{G}$ and $\mathrm{C}_{\mathbf{Q}}(e, M) = \mathrm{C}_{\mathbf{Q}}(e, M_\ell)$. Thus, for every edge $f = (s, p)$ in $\mathrm{D}_{\mathbf{Q}}(e, M)$, $s \succ_H r$. This completes the proof.

We next consider the case where $e \in A_{z-1} \setminus T_r$. Here we prove that $M \cap E_r[i]$ is a base of $\mathbf{Z}_r|D_r[i]$ for every integer i in $[m_r]$. Let i be an integer in $[m_r]$. Since $M(r) = T_r$, $M \cap E_r[i] = D_r[i]$. Since $M(r) \in \mathcal{F}_r$, this and (I1) imply that $D_r[i] \in \mathcal{F}_r$. Thus, $D_r[i]$ is an independent set of $\mathbf{Z}_r|D_r[i]$. Furthermore, for every independent set I of $\mathbf{Z}_r|D_r[i]$, $I \subseteq D_r[i]$. Thus, $D_r[i]$ is a base of $\mathbf{Z}_r|D_r[i]$.

Assume that $e \in E_r^x$. Since $e \notin T_r$, $e \notin D_r^x$. This implies that $\{e\}$ is not an independent set of $\mathbf{Z}_r/D_r[x-1]$. Since $M \cap E_r[x-1]$ is a

Algorithm 1:

1 Define $M_0 := \emptyset$ and $A_0 := E$.
2 Set $t := 1$.
3 **while** $t \le n$ **do**
4   **for** *each resident* r *in* R_t **do**
5   Define $\mathbf{Z}_r := \mathbf{P}_r|A_{t-1}(r)$ and $D_r[0] := \emptyset$.
6   **for** *each integer* i *in* $[m_r]$ **do**
7   Define D_r^i as the set of edges e in $E_r^i \cap A_{t-1}$ such that $\{e\}$ is an independent set of $\mathbf{Z}_r/D_r[i-1]$.
8   Define $D_r[i] := D_r[i-1] \cup D_r^i$.
9   **end**
10   Define $T_r := D_r[m_r]$.
11   **end**
12   **if** *there exists a resident* r *in* R_t *such that* $T_r \notin \mathcal{F}_r$. **then**
13   Output **null**, and halt.
14   **end**
15   Define $F_t := \bigcup_{r \in R_t} T_r$.
16   **if** $M_{t-1} \cup F_t \notin \mathcal{G}$ **then**
17   Output **null**, and halt.
18   **end**
19   Define $M_t := M_{t-1} \cup F_t$.
20   Define L_t as the set of edges (r,h) in A_{t-1} such that $r \notin R[t]$ and $M_t + (r,h) \notin \mathcal{G}$.
21   Define $A_t := A_{t-1} \setminus L_t$.
22   Set $t := t+1$.
23 **end**
24 Output M_n, and halt.

base of $\mathbf{Z}_r|D_r[x-1]$, Lemma 3.5 implies that

$$(M \cap E_r[x-1]) + e \notin \mathcal{F}_r.$$

Thus, $M(r) + e \notin \mathcal{F}_r$. Furthermore, Lemma 3.2 implies that

$$\mathsf{C}_{\mathbf{P}_r}(e, M(r)) = \mathsf{C}_{\mathbf{P}_r}(e, M \cap E_r[x-1]).$$

Thus, for every edge f in $\mathsf{D}_{\mathbf{P}_r}(e, M(r))$, $f \succ_r e$. This completes the proof. ◻

Recall that we assume that Algorithm 1 halts when $t = k$.

Lemma 4.2. *Assume that $k \in [n]$ and there exists a super-stable matching in G. Then for every super-stable matching N in G and every resident r in $R[k]$, $N(r) \subseteq T_r$.*

Proof. An edge (r,h) in E is called a *bad edge* if the following conditions (i) to (iii) are satisfied. (i) $r \in R[k]$. (ii) $(r,h) \notin T_r$. (iii) There exists a super-stable matching N in G such that $(r,h) \in N$. For proving this lemma, it is sufficient to prove that there does not exist a bad edge. We prove this by contradiction. Assume that there exists a bad edge in E. For every bad edge (r,h) in E such that $r \in R_\ell$, one of the following statements holds.

- $(r,h) \notin A_{\ell-1}$, i.e., $(r,h) \in L_z$ for some integer z in $[\ell-1]$.
- $(r,h) \in A_{\ell-1} \setminus T_r$.

We denote by Δ_1 the set of integers ℓ in $[k-1]$ such that there exists a bad edge in L_ℓ. We denote by Δ_2 the set of integers ℓ in $[k]$ such that for some resident r in R_ℓ, there exists a bad edge (r,h) in $A_{\ell-1} \setminus T_r$. Notice that $\Delta_1 \cup \Delta_2 \neq \emptyset$. For each integer i in $\{1,2\}$, we denote by z_i the minimum integer in Δ_i (if $\Delta_i = \emptyset$, then we define $z_i := \infty$).

We first assume that $z_1 < z_2$. Define $z := z_1$. Let $e = (r,h)$ be a bad edge in L_z. Furthermore, let N be a super-stable matching in G such that $e \in N$. Since $z \le k-1$, Lines 16 to 19 of Algorithm 1 imply that $M_z \in \mathcal{G}$. Furthermore, since $e \in L_z$, $M_z + e \notin \mathcal{G}$. Thus, $\mathsf{C}_{\mathbf{Q}}(e, M_z)$ is well-defined. Define $C := \mathsf{C}_{\mathbf{Q}}(e, M_z)$. Then since $C \subseteq N$ contradicts $N \in \mathcal{G}$, $C \setminus N \neq \emptyset$. Since $e \in L_z$, $r \notin R[z]$. Thus, for every edge (s,p) in $C \setminus N$, $s \succ_H r$ follows from $s \in R[z]$. For each edge f in $C \setminus N$ such that $N + f \notin \mathcal{G}$, we define $C_f := \mathsf{C}_{\mathbf{Q}}(f, N)$.

Claim 1. *For any edge f in $C \setminus N$, $N + f \notin \mathcal{G}$. Furthermore, for every pair of edges $f = (s,p)$ in $C \setminus N$ and $(\hat{s}, \hat{p})$ in $C_f - f$, $\hat{s} \succ_H s$.*

Proof. To prove this claim, we prove that for every edge $f = (s,p)$ in $C \setminus N$, $N(s) \subseteq M_z$. Assume that there exists an edge $f = (s,p)$ in $C \setminus N$ such that $N(s) \not\subseteq M_z$. Let g be an edge in $N(s) \setminus M_z$. Since $s \in R[z]$, $s \in R_\ell$ for some integer ℓ in $[z]$. Assume that $g \in A_{\ell-1} \setminus T_s$. Since N is super-stable and $g \in N$, $z_2 \le \ell$. However, this contradicts $z_1 < z_2$. Thus, $g \notin A_{\ell-1} \setminus T_s$. If $g \in T_s$, then since $\ell \le z \le k-1$, $g \in T_s \subseteq M_z$. This contradicts $g \in N(s) \setminus M_z$. Thus, $g \notin A_{\ell-1}$. This implies that there exists an integer j in $[\ell-1]$ such that $g \in L_j$. Since N is super-stable and $g \in N$, g is a bad edge. Thus, $j \in \Delta_1$. Since $j < \ell \le z_1$, this contradicts the fact that z_1 is the minimum integer in Δ_1. This completes the proof.

We are now ready to prove this claim. Assume that there exists an edge $f = (s,p)$ in $C \setminus N$ satisfying one of the following conditions. (i) $N + f \in \mathcal{G}$. (ii) $N + f \notin \mathcal{G}$ and there exists an edge $(\hat{s}, \hat{p})$ in $C_f - f$ such that $s \succsim_H \hat{s}$. Since N is super-stable, $N(s) + f \notin \mathcal{F}_s$. Define $C' := \mathsf{C}_{\mathbf{P}_s}(f, N(s))$. Since $C' - f \subseteq N(s)$ and $N(s) \subseteq M_z$, $C' \subseteq M_z$. However, this contradicts $M_z(s) = T_s \in \mathcal{F}_s$. This completes the proof. ◻

For any edge $f = (s,p)$ in $C \setminus N$, since $e \in N$, $e \neq f$. Thus, for every edge $f = (s,p)$ in $C \setminus N$, since $s \succ_H r$, $e \notin C_f$ follows from Claim 1. For every edge f in $C \setminus N$, $f \in C \cap C_f$. Thus, Lemma 3.3 implies that there exists a circuit C' of $\mathbf{Q}$ such that

$$C' \subseteq (C \cup C^*) \setminus (C \setminus N),$$

where C^* is $\bigcup_{f \in C \setminus N} C_f$. Thus, since $C_f - f \subseteq N$ for every edge f in $C \setminus N$, $C' \subseteq N$. However, this contradicts $N \in \mathcal{G}$. This completes the proof.

We next assume that $z_2 \le z_1$. Define $z := z_2$. Let r be a resident in R_z such that there exists an edge $e = (r,h)$ in $A_{z-1} \setminus T_r$. Let N be a super-stable matching in G such that $e \in N$.

We first prove that $N(r) \subseteq A_{z-1}$. Assume that $N(r) \not\subseteq A_{z-1}$. Let f be an edge in $N(r) \setminus A_{z-1}$. Then there exists an integer ℓ in $[z-1]$ such that $f \in L_\ell$. Since N is super-stable and $f \in N$, $\ell \in \Delta_1$. This implies that $z_1 \le \ell < z_2$. However, this contradicts $z_2 \le z_1$. This completes the proof.

Assume that $e \in E_r^x$. Without loss of generality, we assume that

$$(A_{z-1} \setminus T_r) \cap E_r^i \cap N = \emptyset \tag{1}$$

for every integer i in $[x-1]$. Since $N(r) \subseteq A_{z-1}$, (1) implies $N \cap E_r[i] \subseteq D_r[i]$ for every integer i in $[x-1]$.

Claim 2. *$N \cap E_r[x-1]$ is a base of $\mathbf{Z}_r|D_r[x-1]$.*

Proof. Define $B := N \cap E_r[x-1]$. Since $N(r) \in \mathcal{F}_r$, (I1) implies that B is an independent set of $\mathbf{Z}_r|D_r[x-1]$. Assume that B is not a base of $\mathbf{Z}_r|D_r[x-1]$. Then (I2) implies that there exists an edge f in $D_r[x-1] \setminus N$ such that $B + f \in \mathcal{F}_r$. Assume that $N(r) + f \notin \mathcal{F}_r$. Since $B + f \in \mathcal{F}_r$, $\mathrm{D}_{\mathbf{P}_r}(f, N(r))$ is not a subset of B. For every edge g in $\mathrm{D}_{\mathbf{P}_r}(f, N(r)) \setminus B$, since $f \in E_r[x-1]$ and $g \notin E_r[x-1]$, $f \succ_r g$. Since N is super-stable, $N + f \notin \mathcal{G}$ and $s \succ_H r$ for every edge (s, p) in $\mathrm{D}_{\mathrm{Q}}(f, N)$.

Since $\{f\} \in \mathcal{G}$, $\mathrm{D}_{\mathrm{Q}}(f, N) \neq \emptyset$. Thus, $z \geq 2$. Since $f \in A_{z-1}$ and $A_{z-1} = A_{z-2} \setminus L_{z-1}$, $M_{z-1} + f \in \mathcal{G}$. In what follows, we prove that $\mathrm{D}_{\mathrm{Q}}(f, N) \subseteq M_{z-1}$. This contradicts $M_{z-1} + f \in \mathcal{G}$. Let $g = (s, p)$ be an edge in $\mathrm{D}_{\mathrm{Q}}(f, N)$. Then it is sufficient to prove that $g \in M_{z-1}$. Since $s \succ_H r$, $s \in R_\ell$ for some integer ℓ in $[z-1]$. If we can prove that $N(s) \subseteq T_s$, then $g \in N(s) \subseteq T_s = M_{z-1}(s)$. This completes the proof.

Assume that $N(s) \not\subseteq T_s$. Let $\hat{g}$ be an edge in $N(s) \setminus T_s$. Assume that $\hat{g} \notin A_{\ell-1}$. Then there exists an integer j in $[\ell-1]$ such that $\hat{g} \in L_j$. Since N is super-stable and $\hat{g} \in N$, $j \in \Delta_1$. However, since $z_1 \leq j < \ell < z_2$, this contradicts $z_2 \leq z_1$. Thus, $\hat{g} \in A_{\ell-1}$. Recall that $\hat{g} \notin T_s$. This implies that $\hat{g} \in A_{\ell-1} \setminus T_s$. Since N is super-stable and $\hat{g} \in N$, $\ell \in \Delta_2$. Since $\ell < z_2$, this contradicts the fact that z_2 is the minimum integer in Δ_2. This completes the proof. □

Define
$$B := (N \cap E_r[x-1]) + e.$$
Since $N(r) \in \mathcal{F}_r$, (I1) implies that $B \in \mathcal{F}_r$. Recall that $N(r) \subseteq A_{z-1}$. Thus, B is an independent set of $\mathbf{Z}_r$. Lemma 3.5 and Claim 2 imply that $\{e\}$ is an independent set of $\mathbf{Z}_r/D_r[x-1]$. That is, $e \in D_r^x$. This contradicts $e \notin T_r$. This completes the proof. □

Lemma 4.3. *Assume that $k \in [n]$ and there exists a super-stable matching in G. Then for every super-stable matching N in G and every resident r in $R[k-1]$, $N(r) = T_r$.*

Proof. Lemma 4.2 implies that for every super-stable matching N in G and every resident r in $R[k-1]$, $N(r) \subseteq T_r$. Assume that there exist a super-stable matching N in G and a resident r in $R[k-1]$ such that $N(r) \subsetneq T_r$. Since $T_r, N(r) \in \mathcal{F}_r$ and $|N(r)| < |T_r|$, (I2) implies that there exists an edge e in $T_r \setminus N(r)$ such that $N(r) + e \in \mathcal{F}_r$. Thus, since N is super-stable, $N + e \notin \mathcal{G}$ and $s \succ_H r$ for every edge (s, p) in $\mathrm{D}_{\mathrm{Q}}(e, N)$. Assume that $r \in R_z$. Then since $e \in A_{z-1}$, $M_{z-1} + e \in \mathcal{G}$.

Let $f = (s, p)$ be an edge in $\mathrm{D}_{\mathrm{Q}}(e, N)$. Then since $s \succ_H r$, there exists an integer ℓ in $[z-1]$ such that $s \in R_\ell$. Thus, Lemma 4.2 implies that $N(s) \subseteq T_s$. This implies that $f \in N(s) \subseteq T_s = M_{z-1}(s)$. Thus, $\mathrm{D}_{\mathrm{Q}}(e, N) \subseteq M_{z-1}$. However, this contradicts $M_{z-1} + e \in \mathcal{G}$. This completes the proof. □

Lemma 4.4. *Assume that $k \in [n]$ and there exists a super-stable matching in G. Then for every super-stable matching N in G and every resident r in R_k such that $T_r \in \mathcal{F}_r$, $N(r) = T_r$.*

Proof. Let r be a resident in R_k such that $T_r \in \mathcal{F}_r$. Notice that Lemma 4.2 implies that $N(r) \subseteq T_r$ for every super-stable matching N in G. Assume that there exist a super-stable matching N in G such that $N(r) \subsetneq T_r$. Since $T_r, N(r) \in \mathcal{F}_r$ and $|N(r)| < |T_r|$, (I2) implies that there exists an edge e in $T_r \setminus N(r)$ such that $N(r) + e \in \mathcal{F}_r$. Thus, since N is super-stable, $N + e \notin \mathcal{G}$ and $s \succ_H r$ for every edge (s, p) in $\mathrm{D}_{\mathrm{Q}}(e, N)$. Since $e \in A_{k-1}$, $M_{k-1} + e \in \mathcal{G}$.

Let $f = (s, p)$ be an edge in $\mathrm{D}_{\mathrm{Q}}(e, N)$. Then since $s \succ_H r$, there exists an integer ℓ in $[k-1]$ such that $s \in R_\ell$. Thus, Lemma 4.2 implies that $N(s) \subseteq T_s$. This implies that $f \in N(s) \subseteq T_s = M_{k-1}(s)$. Thus, $\mathrm{D}_{\mathrm{Q}}(e, N) \subseteq M_{k-1}$. However, this contradicts $M_{k-1} + e \in \mathcal{G}$. This completes the proof. □

Lemma 4.5. *If Algorithm 1 outputs* **null***, then there does not exist a super-stable matching in G.*

Proof. Notice that in this case, $k \in [n]$. We prove this lemma by contradiction. Assume that there exists a super-stable matching N in G.

We first assume that Algorithm 1 outputs **null** at Line 13. Since Algorithm 1 outputs **null** when $t = k$, there exists a resident r in R_k such that $T_r \notin \mathcal{F}_r$. Recall that Lemma 4.2 implies that $N(r) \subseteq T_r$. Thus, since $T_r \notin \mathcal{F}_r$ and $N(r) \in \mathcal{F}_r$, $N(r) \subsetneq T_r$.

Since $N(r) \subseteq T_r$, $N \cap E_r^i \subseteq D_r^i$ for every integer i in $[m_r]$. Let x be the minimum integer in $[m_r]$ such that $N \cap E_r^x \subsetneq D_r^x$. Let e be an edge in $D_r^x \setminus N$. Notice that $N \cap E_r^i = D_r^i$ for every integer i in $[x-1]$. Since $N(r) \in \mathcal{F}_r$, (I1) implies that
$$N \cap E_r[x-1] = D_r[x-1]$$
is an independent set of $\mathbf{Z}_r|D_r[x-1]$. Thus, since $I \subseteq D_r[x-1]$ for every independent set I of $\mathbf{Z}_r|D_r[x-1]$, $N \cap E_r[x-1]$ is a base of $\mathbf{Z}_r|D_r[x-1]$. This and $e \in D_r^x$ imply that
$$(N \cap E_r[x-1]) + e \in \mathcal{F}_r.$$
This implies that if $N(r) + e \notin \mathcal{F}_r$, then $\mathrm{D}_{\mathbf{P}_r}(e, N(r))$ is not a subset of $E_r[x-1]$. Furthermore, since $e \in E_r^x$, $e \succsim_r f$ for every edge f in $\mathrm{D}_{\mathbf{P}_r}(e, N(r)) \setminus E_r[x-1]$. Since N is super-stable, $N + e \notin \mathcal{G}$ and $s \succ_H r$ for every edge (s, p) in $\mathrm{D}_{\mathrm{Q}}(e, N)$.

Since $e \in T_r$, $e \in A_{k-1}$. Thus, $M_{k-1} + e \in \mathcal{G}$. On the other hand, Lemma 4.3 implies that $N(s) = T_s = M_{k-1}(s)$ for every resident s in $R[k-1]$. Furthermore, since $r \in R_k$, $s \in R[k-1]$ for every edge (s, p) in $\mathrm{D}_{\mathrm{Q}}(e, N)$. Thus, $\mathrm{C}_{\mathrm{Q}}(e, N)$ is a subset of $M_{k-1} + e$. However, this contradicts $M_{k-1} + e \in \mathcal{G}$. This completes the proof.

We next assume that Algorithm 1 outputs **null** at Line 17. Assume that there exists a super-stable matching N in G. Lines 12 and 13 of Algorithm 1 imply that $T_r \in \mathcal{F}_r$ for every resident r in $R[k]$. Thus, Lemmas 4.3 and 4.4 imply that $M_{k-1} \cup F_k \subseteq N$. Since Algorithm 1 outputs **null** at Line 17, $M_{k-1} \cup F_k \notin \mathcal{G}$. Thus, $N \notin \mathcal{G}$. This contradicts $N \in \mathcal{G}$. This completes the proof. □

Theorem 4.6. *Algorithm 1 can solve the super-stable matching problem.*

Proof. This theorem follows from Lemmas 4.1 and 4.5. □

5 STRONGLY STABLE MATCHINGS

In this section, we propose an algorithm for the strongly stable matching problem (see Algorithm 2). This algorithm is based on the algorithm of [14] for the strongly stable matching problem in the many-to-one setting with matroid constraints. For proving that Algorithm 2 is a polynomial-time algorithm, it is sufficient to prove that we can decide whether each subset of E is an independent set of the matroids in Algorithm 2 in time bounded by a polynomial in the input size of G. We can easily prove this by Lemma 3.5 as Algorithm 1. By using the algorithm of [3] at Line 19 of Algorithm 2, we can implement Algorithm 2 in $O(|E|^{3.5}\mathsf{EO})$ time, where EO is the time complexity of the independence oracle for the given matroids.

Algorithm 2:

1 Define $M_0 := \emptyset$, $A_0 := E$, and $F[0] := \emptyset$.
2 Set $t := 1$.
3 **while** $t \le n$ **do**
4   **for** *each resident* r *in* R_t **do**
5   Define $\mathbf{Z}_r := \mathbf{P}_r | A_{t-1}(r)$ and $D_r[0] := \emptyset$.
6   **for** *each integer* i *in* $[m_r]$ **do**
7   Define D_r^i as the set of edges e in $E_r^i \cap A_{t-1}$ such that $\{e\}$ is an independent set of $\mathbf{Z}_r / D_r[i-1]$.
8   Define $D_r[i] := D_r[i-1] \cup D_r^i$.
9   Define $\mathbf{Z}_r^i := (\mathbf{Z}_r / D_r[i-1]) | D_r^i$.
10   **end**
11   Define $T_r := D_r[m_r]$ and $\mathbf{S}_r := \bigoplus_{i \in [m_r]} \mathbf{Z}_r^i$.
12   **end**
13   Define $F_t := \bigcup_{r \in R_t} T_r$ and $F[t] := F[t-1] \cup F_t$.
14   Define $\mathbf{Q}_t := (\mathbf{Q}/F[t-1]) | F_t$.
15   Define $\sigma_t := \sum_{r \in R_t} \mathbf{r}(\mathbf{S}_r)$.
16   **if** $\mathbf{r}(\mathbf{Q}_t) > \sigma_t$ **then**
17   Output **null**, and halt.
18   **end**
19   Find a maximum-size common independent set I_t of $\bigoplus_{r \in R_t} \mathbf{S}_r$ and $\mathbf{Q}_t$.
20   **if** $|I_t| < \sigma_t$ **then**
21   Output **null**, and halt.
22   **end**
23   Define $M_t := M_{t-1} \cup I_t$.
24   Define L_t as the set of edges (r,h) in A_{t-1} such that $r \notin R[t]$ and $M_t + (r,h) \notin \mathcal{G}$.
25   Define $A_t := A_{t-1} \setminus L_t$.
26   Set $t := t+1$.
27 **end**
28 Output M_n, and halt.

What remains is to prove the correctness of Algorithm 2. In the rest of this section, we assume that Algorithm 2 halts when $t = k$.

Lemma 5.1. *For every integer ℓ in $[k-1]$, every resident r in R_ℓ, and every integer i in $[m_r]$, $I_\ell \cap D_r[i]$ is a base of $\mathbf{Z}_r | D_r[i]$.*

Proof. Let ℓ be an integer in $[k-1]$. Since I_ℓ is an independent set of $\bigoplus_{r \in R_\ell} \mathbf{S}_r$, $|I_\ell| \le \sigma_\ell$. Lines 20 and 21 of Algorithm 2 imply that $\sigma_\ell \le |I_\ell|$. Thus, $|I_\ell| = \sigma_\ell$. Furthermore, since I_ℓ is an independent set of $\bigoplus_{r \in R_\ell} \mathbf{S}_r$, $I_\ell(r)$ is an independent set of $\mathbf{S}_r$ for every resident r in R_ℓ. Thus, $|I_\ell(r)| = \mathbf{r}(\mathbf{S}_r)$ for every resident r in R_ℓ. This implies that $I_\ell(r)$ is a base of $\mathbf{S}_r$ for every resident r in R_ℓ. Thus, $I_\ell \cap D_r^i$ is a base of $\mathbf{Z}_r^i$ for every resident r in R_ℓ and every integer i in $[m_r]$.

Let r be a resident in R_ℓ. Let x be an integer in $[m_r]$. Assume that $I_\ell \cap D_r[x-1]$ is a base of $\mathbf{Z}_r | D_r[x-1]$. (That is, if $x = 1$, then we make no assumption.) Notice that

$$\mathbf{Z}_r | D_r[x-1] = (\mathbf{Z}_r | D_r[x]) | D_r[x-1].$$

Thus, since Lemma 3.4 implies that

$$\mathbf{Z}_r^x = (\mathbf{Z}_r | D_r[x]) / D_r[x-1]$$

and $I_\ell \cap D_r^x$ is a base of $\mathbf{Z}_r^x$, Lemma 3.5 imply that $I_\ell \cap D_r[x]$ is a base of $\mathbf{Z}_r | D_r[x]$. This completes the proof. ◻

Lemma 5.2. *For every integer ℓ in $[k-1]$, M_ℓ is a base of $\mathbf{Q} | F[\ell]$.*

Proof. Let ℓ be an integer in $[k-1]$. Since I_ℓ is an independent set of $\mathbf{Q}_\ell$, $|I_\ell| \le \mathbf{r}(\mathbf{Q}_\ell)$. Furthermore, Lines 16 and 20 of Algorithm 2 imply that $\mathbf{r}(\mathbf{Q}_\ell) \le \sigma_\ell$ and $\sigma_\ell \le |I_\ell|$. Thus, $|I_\ell| = \mathbf{r}(\mathbf{Q}_\ell)$. Since I_ℓ is an independent set of $\mathbf{Q}_\ell$, this implies that I_ℓ is a base of $\mathbf{Q}_\ell$.

Assume that $M_{\ell-1}$ is a base of $\mathbf{Q} | F[\ell-1]$. (That is, if $\ell = 1$, then we make no assumption.) Notice that

$$\mathbf{Q} | F[\ell-1] = (\mathbf{Q} | F[\ell]) | F[\ell-1].$$

Since Lemma 3.4 implies that

$$(\mathbf{Q}/F[\ell-1]) | F_\ell = (\mathbf{Q} | F[\ell]) / F[\ell-1]$$

and I_ℓ is a base of $\mathbf{Q}_\ell = (\mathbf{Q}/F[\ell-1]) | F_\ell$, Lemma 3.5 implies that $M_{\ell-1} \cup I_\ell$ is a base of $\mathbf{Q} | F[\ell]$. This completes the proof. ◻

Lemma 5.3. *If Algorithm 2 outputs M_n, then M_n is a strongly stable matching in G.*

Proof. Notice that in this case, $k = n+1$. Define $M := M_n$. We first prove that for every resident r in R, $M_n(r) \in \mathcal{F}_r$. For proving this, it suffices to prove that for every integer ℓ in $[n]$ and every resident r in R_ℓ, $I_\ell(r) \in \mathcal{F}_r$. Lemma 5.1 implies that for every integer ℓ in $[n]$ and every resident r in R_ℓ, $I_\ell \cap D_r[m_r]$ is a base of $\mathbf{Z}_r | D_r[m_r]$. Thus, for every integer ℓ in $[n]$ and every resident r in R_ℓ, since $I_\ell(r) \subseteq T_r$, $I_\ell(r) \in \mathcal{F}_r$. Furthermore, Lemma 5.2 implies that $M \in \mathcal{G}$. Thus, M is a matching in G. What remains is to prove that M is strongly stable. Let $e = (r,h)$ be an edge in $E \setminus M$. Assume that $r \in R_z$. Let x be the integer in $[m_r]$ such that $e \in E_r^x$.

We first assume that $e \notin A_{z-1}$. Then $e \in L_\ell$ for some integer ℓ in $[z-1]$. Thus, $M_\ell + e \notin \mathcal{G}$ and $s \succ_H r$ for every edge (s,p) in $\mathsf{D}_{\mathbf{Q}}(e, M_\ell)$. Since $M_\ell \subseteq M$, Lemma 3.2 implies that $M + e \notin \mathcal{G}$ and $\mathsf{C}_{\mathbf{Q}}(e, M) = \mathsf{C}_{\mathbf{Q}}(e, M_\ell)$. Thus, for every edge $f = (s,p)$ in $\mathsf{D}_{\mathbf{Q}}(e, M)$, $s \succ_H r$. This completes the proof.

We next assume that $e \in A_{z-1} \setminus T_r$. In this case, $e \notin D_r^x$. This implies that $\{e\}$ is not an independent set of $\mathbf{Z}_r / D_r[x-1]$. Since $M \cap E_r[x-1] = I_z \cap D_r[x-1]$, Lemma 3.5 and Lemma 5.1 imply that

$$(M \cap E_r[x-1]) + e \notin \mathcal{F}_r.$$

Thus, $M(r) + e \notin \mathcal{F}_r$. Furthermore, Lemma 3.2 implies that

$$\mathsf{C}_{\mathbf{P}_r}(e, M(r)) = \mathsf{C}_{\mathbf{P}_r}(e, M \cap E_r[x-1]).$$

Thus, for every edge f in $\mathsf{D}_{\mathbf{P}_r}(e, M(r))$, $f \succ_r e$. This completes the proof.

Lastly, we consider the case where $e \in T_r$, i.e., $e \in D_r^x$. Lemma 5.1 implies that $M \cap E_r[x]$ is a base of $\mathbf{Z}_r | D_r[x]$. This implies that

$$(M \cap E_r[x]) + e \notin \mathcal{F}_r.$$

Thus, $M(r) + e \notin \mathcal{F}_r$. Furthermore, Lemma 3.2 implies that

$$\mathsf{C}_{\mathbf{P}_r}(e, M(r)) = \mathsf{C}_{\mathbf{P}_r}(e, M \cap E_r[x]).$$

This implies that for every edge f in $\mathsf{D}_{\mathbf{P}_r}(e, M(r))$, $f \succsim_r e$. What remains is to prove that $M + e \notin \mathcal{G}$ and for every edge $f = (s,p)$ in $\mathsf{D}_{\mathbf{Q}}(e, M)$, $s \succsim_H r$.

Since Lemma 5.2 implies that M_z is a base of $\mathbf{Q}|F[z]$ and $e \in F_z$, $M_z + e \notin \mathcal{G}$. Thus, $M + e \notin \mathcal{G}$. Lemma 3.2 implies that $\mathrm{C_Q}(e, M) = \mathrm{C_Q}(e, M_z)$. Thus, for every edge (s, p) in $\mathrm{D_Q}(e, M)$, $s \in R[z]$. This completes the proof. □

Recall that we assume that Algorithm 2 halts when $t = k$.

Lemma 5.4. *Assume that $k \in [n]$ and there exists a strongly stable matching in G. Then for every strongly stable matching N in G and every resident r in $R[k]$, $N(r) \subseteq T_r$.*

Proof. An edge (r, h) in E is called a *bad edge* if the following conditions (i) to (iii) are satisfied. (i) $r \in R[k]$. (ii) $(r, h) \notin T_r$. (iii) There exists a strongly stable matching N in G such that $(r, h) \in N$. For proving this lemma, it is sufficient to prove that there does not exist a bad edge. We prove this by contradiction. Assume that there exists a bad edge in E. For every bad edge (r, h) in E such that $r \in R_\ell$, one of the following statements holds.

- $(r, h) \notin A_{\ell-1}$, i.e., $(r, h) \in L_z$ for some integer z in $[\ell - 1]$.
- $(r, h) \in A_{\ell-1} \setminus T_r$.

We denote by Δ_1 the set of integers ℓ in $[k - 1]$ such that there exists a bad edge in L_ℓ. We denote by Δ_2 the set of integers ℓ in $[k]$ such that for some resident r in R_ℓ, there exists a bad edge (r, h) in $A_{\ell-1} \setminus T_r$. Notice that $\Delta_1 \cup \Delta_2 \neq \emptyset$. For each integer i in $\{1, 2\}$, we denote by z_i the minimum integer in Δ_i (if $\Delta_i = \emptyset$, then we define $z_i := \infty$).

We first consider the case where $z_1 < z_2$. Define $z := z_1$. Let $e = (r, h)$ be a bad edge in L_z. Furthermore, let N be a strongly stable matching in G such that $e \in N$. Since $z \leq k - 1$, Lemma 5.2 implies that $M_z \in \mathcal{G}$. Furthermore, since $e \in L_z$, $M_z + e \notin \mathcal{G}$. Thus, $\mathrm{C_Q}(e, M_z)$ is well-defined. Define $C := \mathrm{C_Q}(e, M_z)$. Then since $C \subseteq N$ contradicts $N \in \mathcal{G}$, $C \setminus N \neq \emptyset$. Since $e \in L_z$, $r \notin R[z]$. Thus, for every edge (s, p) in $C \setminus N$, $s \succ_H r$ follows from $s \in R[z]$. For each edge f in $C \setminus N$ such that $N + f \notin \mathcal{G}$, we define $C_f := \mathrm{C_Q}(f, N)$.

Claim 3. *For every edge $f = (s, p)$ in $C \setminus N$ such that $N(s) + f \notin \mathcal{F}_s$, there exists an edge g in $\mathrm{D}_{\mathbf{P}_s}(f, N(s))$ such that $f \succsim_s g$.*

Proof. We first prove that $N(s) \subseteq T_s$ for every edge $f = (s, p)$ in $C \setminus N$. Assume that there exists an edge $f = (s, p)$ in $C \setminus N$ such that $N(s) \not\subseteq T_s$. Let g be an edge in $N(s) \setminus T_s$. Since $s \in R[z]$, $s \in R_\ell$ for some integer ℓ in $[z]$. Assume that $g \in A_{\ell-1} \setminus T_s$. Then since N is strongly stable and $g \in N$, $z_2 \leq \ell$. However, this contradicts $z_1 < z_2$. Thus, $g \notin A_{\ell-1} \setminus T_s$. Since $g \notin T_s$, $g \notin A_{\ell-1}$. This implies that there exists an integer j in $[\ell - 1]$ such that $g \in L_j$. Since N is strongly stable and $g \in N$, g is a bad edge. Thus, $j \in \Delta_1$. However, since $j < \ell \leq z_1$, this contradicts the fact that z_1 is the minimum integer in Δ_1. This completes the proof.

We now ready to prove this claim. Let $f = (s, p)$ be an edge in $C \setminus N$ such that $N(s) + f \notin \mathcal{F}_s$. Assume that $f \in E_s^x$. Then since $f \in C$, $f \in T_s$. Thus, $f \in D_s^x$. Since $N(s) \subseteq T_s$,

$$N \cap E_s[x - 1] \subseteq D_s[x - 1].$$

Thus, since (I1) implies that $N \cap E_s[x - 1]$ is an independent set of $\mathbf{Z}_r|D_s[x - 1]$, (I2) implies that there exists a base B of $\mathbf{Z}_r|D_s[x - 1]$ such that $N \cap E_s[x - 1] \subseteq B$. Since $f \in D_s^x$, Lemma 3.5 implies that $B + f \in \mathcal{F}_s$. This and (I1) imply that

$$(N \cap E_s[x - 1]) + f \in \mathcal{F}_s.$$

Thus, $\mathrm{D}_{\mathbf{P}_s}(f, N(s))$ is not a subset of $E_s[x - 1]$. Let g be an edge in $\mathrm{D}_{\mathbf{P}_s}(f, N(s)) \setminus E_s[x - 1]$. Since $f \in E_s^x$ and $g \notin E_s[x - 1]$, $f \succsim_s g$. This completes the proof. □

Claim 4. *For any edge f in $C \setminus N$, $N + f \notin \mathcal{G}$. Furthermore, for every pair of edges $f = (s, p)$ in $C \setminus N$ and $(\hat{s}, \hat{p})$ in $C_f - f$, $\hat{s} \succsim_H s$.*

Proof. Assume that there exists an edge $f = (s, p)$ in $C \setminus N$ such that one of the following conditions is satisfied. (i) $N + f \in \mathcal{G}$. (ii) $N + f \notin \mathcal{G}$ and there exists an edge $(\hat{s}, \hat{p})$ in $C_f - f$ such that $s \succ_H \hat{s}$. Since N is strongly stable, $N(s) + f \notin \mathcal{F}_s$. Claim 3 implies that there exists an edge g in $\mathrm{D}_{\mathbf{P}_s}(f, N(s))$ such that $f \succsim_s g$. This contradicts that fact that N is strongly stable. This completes the proof. □

For any edge $f = (s, p)$ in $C \setminus N$, since $e \in N$, $e \neq f$. Thus, for every edge $f = (s, p)$ in $C \setminus N$, since $s \succ_H r$, $e \notin C_f$ follows from Claim 4. For every edge f in $C \setminus N$, $f \in C \cap C_f$. Thus, Lemma 3.3 implies that there exists a circuit C' of $\mathbf{Q}$ such that

$$C' \subseteq (C \cup C^*) \setminus (C \setminus N),$$

where C^* is $\bigcup_{f \in C \setminus N} C_f$. Thus, since $C_f - f \subseteq N$ for every edge f in $C \setminus N$, C' is a subset of N. This contradicts the fact that $N \in \mathcal{G}$. This completes the proof.

We next consider the case where $z_2 \leq z_1$. Define $z := z_2$. Let r be a resident in R_z such that there exists an edge $e = (r, h)$ in $A_{z-1} \setminus T_r$. Let N be a strongly stable matching in G such that $e \in N$.

Here we prove that $N(s) \subseteq A_{\ell-1}$ for every integer ℓ in $[z]$ and every resident s in R_ℓ. Assume that there exist an integer ℓ in $[z]$ and a resident s in R_ℓ such that $N(s) \not\subseteq A_{\ell-1}$. Furthermore, let f be an edge in $N(s) \setminus A_{\ell-1}$. Then there exists an integer j in $[\ell - 1]$ such that $f \in L_j$. Since N is strongly stable and $f \in N$, $j \in \Delta_1$. This implies that $z_1 \leq j < \ell \leq z_2$. This contradicts $z_2 \leq z_1$.

We next prove that $N(s) \subseteq T_s$ for every integer ℓ in $[z - 1]$ and every resident s in R_ℓ. Assume that there exist an integer ℓ in $[z-1]$ and a resident s in R_ℓ such that $N(s) \not\subseteq T_s$. Furthermore, let f be an edge in $N(s) \setminus T_s$. Then since $N(s) \subseteq A_{\ell-1}$, $f \in A_{\ell-1} \setminus T_s$. Since N is strongly stable and $f \in N$, $\ell \in \Delta_2$. This contradicts the fact that z_2 is the minimum integer in Δ_2. This completes the proof.

Assume that $e \in E_r^x$. Without loss of generality, we assume that

$$(A_{z-1} \setminus T_r) \cap E_r^i \cap N = \emptyset \tag{2}$$

for every integer i in $[x - 1]$. Since $N(r) \subseteq A_{z-1}$, (2) implies $N \cap E_r[i] \subseteq D_r[i]$ for every integer i in $[x - 1]$.

Claim 5. *$N \cap E_r[x - 1]$ is a base of $\mathbf{Z}_r|D_r[x - 1]$.*

Proof. We prove this claim by contradiction. Define $B := N \cap E_r[x - 1]$. Since $N(r) \in \mathcal{F}_r$, (I1) implies that B is an independent set of $\mathbf{Z}_r|D_r[x-1]$. Assume that B is not a base of $\mathbf{Z}_r|D_r[x-1]$. Then (I2) implies that there exists an edge f in $D_r[x-1] \setminus N$ such that $B + f \in \mathcal{F}_r$. Assume that $N(r) + f \notin \mathcal{F}_r$. Since $B + f \in \mathcal{F}_r$, $\mathrm{D}_{\mathbf{P}_r}(f, N(r))$ is not a subset of B. For every edge g in $\mathrm{D}_{\mathbf{P}_r}(f, N(r)) \setminus B$, since $f \in E_r[x - 1]$ and $g \notin E_r[x - 1]$, $f \succ_r g$. Thus, since N is strongly stable, $N + f \notin \mathcal{G}$.

Since $f \in A_{z-1}$, $M_{z-1} + f \in \mathcal{G}$. Thus, since Lemma 5.2 implies that M_{z-1} is a base of $\mathbf{Q}|F[z - 1]$, Lemma 3.5 implies that $\{f\}$ is an independent set of $\mathbf{Q}/F[z - 1]$. Furthermore, since (I1) implies that $N \cap F[z - 1]$ is an independent set of $\mathbf{Q}|F[z - 1]$, (I2) implies that there exists a base $\hat{B}$ of $\mathbf{Q}|F[z-1]$ such that $N \cap F[z-1] \subseteq \hat{B}$. Since Lemma 3.5 implies that $\hat{B} + f \in \mathcal{G}$, (I1) implies that

$$(N \cap F[z-1]) + f \in \mathcal{G}.$$

This implies that there exists an edge (s,p) in $\mathrm{D}_{\mathbf{Q}}(f,N) \setminus F[z-1]$. Assume that $s \in R_\ell$. If $\ell \leq z-1$, then since $N(s) \subseteq T_s$, $(s,p) \in F_\ell$. This contradicts $(s,p) \notin F[z-1]$. Thus, $s \notin R[z-1]$. Since $r \in R_z$, $r \succsim_H s$. However, this contradicts the fact that N is strongly stable. This completes the proof. □

Define

$$B := (N \cap E_r[x-1]) + e.$$

Since $N(r) \in \mathcal{F}_r$, (I1) implies that $B \in \mathcal{F}_r$. Recall that $N(r) \subseteq A_{z-1}$. Thus, B is an independent set of $\mathbf{Z}_r$. Lemma 3.5 and Claim 5 imply that $\{e\}$ is an independent set of $\mathbf{Z}_r/D_r[x-1]$, i.e., $e \in D_r^x$. This contradicts $e \notin T_r$. This completes the proof. □

Lemma 5.5. *Assume that $k \in [n]$ and there exists a strongly stable matching in G. Then for every strongly stable matching N in G, every resident r in $R[k]$, and every integer i in $[m_r]$, $N \cap E_r^i$ is a base of $\mathbf{Z}_r^i$.*

Proof. Assume that we are given a strongly stable matching N in G, a resident r in $R[k]$, and an integer x in $[m_r]$. Furthermore, we assume that for every integer i in $[x-1]$, $N \cap E_r^i$ is a base of $\mathbf{Z}_r^x$. (That is, if $x = 1$, then we make no assumption.) Then Lemmas 3.4 and 3.5 imply that $N \cap E_r[x-1]$ is a base of $\mathbf{Z}_r|D_r[x-1]$. Since $N(r) \in \mathcal{F}_r$ and $N(r) \subseteq T_r$ follows from Lemma 5.4, (I1) implies that $N \cap E_r[x]$ is an independent set of $\mathbf{Z}_r|D_r[x]$. Thus, Lemmas 3.4 and 3.5 imply that $N \cap E_r^x$ is an independent set of $\mathbf{Z}_r^x$. Assume that $N \cap E_r^x$ is not a base of $\mathbf{Z}_r^x$. Then (I2) implies that there exists an edge e in $D_r^x \setminus N$ such that $(N \cap E_r^x) + e$ is an independent set of $\mathbf{Z}_r^x$. Lemmas 3.4 and 3.5 imply that $(N \cap E_r[x]) + e$ is an independent set of $\mathbf{Z}_r|D_r[x]$. Thus, if $N(r) + e \notin \mathcal{F}_r$, then there exists an edge f in $\mathrm{D}_{\mathbf{P}_r}(e, N(r))$ such that $e \succ_r f$. Thus, since N is strongly stable, $N + e \notin \mathcal{G}$.

Assume that $r \in R_z$. Since $e \in A_{z-1}$, $M_{z-1} + e \in \mathcal{G}$. Thus, since Lemma 5.2 implies that M_{z-1} is a base of $\mathbf{Q}|F[z-1]$, Lemma 3.5 implies that $\{e\}$ is an independent set of $\mathbf{Q}/F[z-1]$. Furthermore, since (I1) implies that $N \cap F[z-1]$ is an independent set of $\mathbf{Q}|F[z-1]$, (I2) implies that there exists a base B of $\mathbf{Q}|F[z-1]$ such that $N \cap F[z-1] \subseteq B$. Lemma 3.5 implies that $B + e \in \mathcal{G}$. Thus, (I1) implies that

$$(N \cap F[z-1]) + e \in \mathcal{G}.$$

Thus, there exists an edge (s,p) in $\mathrm{D}_{\mathbf{Q}}(e,N)$ such that $(s,p) \notin F[z-1]$. Assume that $s \in R_\ell$. If $\ell \leq z-1$, then since $N(s) \subseteq T_s$ follows from Lemma 5.4, $(s,p) \in F_\ell$. This contradicts $(s,p) \notin F[z-1]$. Thus, $s \notin R[z-1]$. Since $r \in R_z$, $r \succsim_H s$. However, this contradicts the fact that N is strongly stable. This completes the proof. □

Lemma 5.6. *Assume that $k \in [n]$ and there exists a strongly stable matching in G. Then for every strongly stable matching N in G and every integer ℓ in $[k]$, $N \cap F_\ell$ is a base of $\mathbf{Q}_\ell$.*

Proof. Assume that we are given a strongly stable matching N in G and an integer ℓ in $[k]$. Furthermore, we assume that $N \cap F[\ell-1]$ is a base of $\mathbf{Q}|F[\ell-1]$. (That is, if $\ell = 1$, then we make no assumption.) Notice that (I1) implies that $N \cap F[\ell]$ is an independent set of $\mathbf{Q}|F[\ell]$. Thus, Lemmas 3.4 and 3.5 imply that $N \cap F_\ell$ is an independent set of $\mathbf{Q}_\ell$. Assume that $N \cap F_\ell$ is not a base of $\mathbf{Q}_\ell$. Then Lemma 3.5 implies that $N \cap F[\ell]$ is not a base of $\mathbf{Q}|F[\ell]$. Thus, (I2) implies that there exists an edge $e = (r,h)$ in $F[\ell] \setminus N$ such that $(N \cap F[\ell]) + e \in \mathcal{G}$. Thus, if $N + e \notin \mathcal{G}$, then $\mathrm{D}_{\mathbf{Q}}(e,N) \not\subseteq F[\ell]$. Let $f = (s,p)$ be an edge in $\mathrm{D}_{\mathbf{Q}}(e,N) \setminus F[\ell]$. Then Lemma 5.4 implies that $s \notin R[\ell]$. Since $r \in R[\ell]$, this implies that $r \succ_H s$. Thus, since N is strongly stable, this implies that $N(r) + e \notin \mathcal{F}_r$.

Assume that $e \in E_r^x$. Define $B := N \cap E_r[x-1]$. It follows from Lemma 5.4 that $B \subseteq D_r[x-1]$. Furthermore, since $N(r) \in \mathcal{F}_r$, (I1) implies that $B \in \mathcal{F}_r$. Thus, (I2) implies that there exists a base of $\hat{B}$ of $\mathbf{Z}_r|D_r[x-1]$ such that $B \subseteq \hat{B}$. Since $e \in E_r^x \cap F[\ell] = D_r^x$, it follows from Lemma 3.5 that $\hat{B} + e \in \mathcal{F}_r$. Thus, (I1) implies that $B + e \in \mathcal{F}_r$. This implies that $\mathrm{D}_{\mathbf{P}_r}(e, N(r)) \not\subseteq B$. Thus, there exists an edge f in $\mathrm{D}_{\mathbf{P}_r}(e, N(r))$ such that $f \notin E_r[x-1]$. This implies that since $e \in E_r^x$, $e \succsim_r f$. However, this contradicts the fact that N is strongly stable. This completes the proof. □

Lemma 5.7. *If Algorithm 2 outputs* **null**, *then there does not exist a strongly stable matching in G.*

Proof. Notice that in this case, $k \in [n]$. We first consider the case where Algorithm 2 outputs **null** at Line 17. That is, $\mathbf{r}(\mathbf{Q}_k) > \sigma_k$. Assume that there exists a strongly stable matching N in G. Lemma 5.6 implies that $N \cap F_k$ is a base of $\mathbf{Q}_k$. Thus, $|N \cap F_k| = \mathbf{r}(\mathbf{Q}_k)$. On the other hand, for every resident r in R_k and every integer i in $[m_r]$, Lemma 5.5 implies that $N \cap E_r^i$ is an independent set of $\mathbf{Z}_r^i$. This implies that for every resident r in R_k, $N(r)$ is an independent set of $\mathbf{S}_r$. Thus, (I1) implies that for every resident r in R_k, $N(r) \cap F_k$ is an independent set of $\mathbf{S}_r$. Thus, $|N \cap F_k| \leq \sigma_k$. This contradicts $\mathbf{r}(\mathbf{Q}_k) > \sigma_k$.

We next assume that Algorithm 2 outputs **null** at Line 21. That is, $|I_k| < \sigma_k$. Assume that there exists a strongly stable matching N in G. Lemma 5.5 implies that for every resident r in R_k, $N(r)$ is a base of $\mathbf{S}_r$. Furthermore, Lemma 5.4 implies that for every resident r in R_k, $N(r) = N(r) \cap F_k$. Thus, $|N \cap F_k| = \sigma_k$. On the other hand, since Lemma 5.6 implies that $N \cap F_k$ is a common independent set of $\bigoplus_{r \in R_k} \mathbf{S}_r$ and $\mathbf{Q}_k$, $|N \cap F_k| \leq |I_k|$. This contradicts $|I_k| < \sigma_k$. This completes the proof. □

Theorem 5.8. *Algorithm 2 can solve the strongly stable matching problem.*

Proof. This theorem follows from Lemmas 5.3 and 5.7. □

6 CONCLUSION

In this paper, we consider the problem of finding a many-to-many super-stable matching and a many-to-many strongly stable matching with master preference lists and matroid constraints, and we prove that these problems can be solved in polynomial time. It is interesting to clarify whether the results in this paper can be extended to the general preference list case.

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