

# Computing Nash Equilibria for District-based Nominations

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## ABSTRACT

We study political parties that strategically place their candidates in districts so to maximise the number of their nominees that get elected. In each district, voters rank the nominated candidates and elect the plurality winners. After studying equilibrium existence in restricted instances, we show that deciding the existence of pure Nash equilibria for these games is NP-complete if party size is bounded by a constant and  $\Sigma_2^P$ -complete for the general case. For the hardness part of the latter result we reduce from  $\exists\exists\exists\text{-3SAT}$ .

## KEYWORDS

Voting; Coalitional Strategies; Nash Equilibria; Complexity

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## 1 INTRODUCTION

In district-based elections, common in the UK and the US, parties put forward candidates to represent subset of voters belonging to certain districts or constituencies, with candidates chosen locally by the voters, typically using plurality-based rules such as First-Past-The-Post. These elections pose a number of distinct algorithmic challenges when compared to those based on proportional representation. Think of the problem of gerrymandering and how governments can strategically manipulate elections by simply redrafting borders, or how to guarantee a good level of overall representation to voters as a whole, despite seats being won locally.

Decision-making challenges in district-based elections are not appearing at the societal level only, but the parties themselves face these continuously. When parties need to choose which candidates to put forward, their decision is going to be based on a number of strategic considerations, including the voters' preferences and the choices of the opposing parties. Some candidates may be fielded by a party even if when they stand no chance of being elected, as their role is to take away votes from other candidates and favour their party indirectly. Similarly, it may be better not to run at all in a district, as candidates may be better employed elsewhere. The next example presents some of the intricacies of strategic nomination in district-based election in a simple setup.

**EXAMPLE 1.** Consider the district in the left side of Figure 1. where three parties,  $A = \{a_1, a_2, a_3, a_4\}$ ,  $B = \{b\}$ , and  $C = \{c\}$ , are competing to elect the plurality winner. The district, let us call it  $D_L$ , consists

5	3	3	4	3	5	3	3	4	3
$a_1$	$a_2$	$a_3$	$b$	$c$	$a_4$	$a_2$	$a_3$	$b$	$c$
$b$	$b$	$c$	$c$	$b$	$b$	$b$	$c$	$c$	$b$
$c$	$a_1$	$a_1$	$a_1$	$a_1$	$c$	$a_4$	$a_4$	$a_4$	$a_4$
$a_2$	$c$	$b$	$a_2$	$a_2$	$a_2$	$c$	$b$	$a_2$	$a_2$
$a_3$	$a_3$	$a_2$	$a_3$	$a_3$	$a_3$	$a_3$	$a_2$	$a_3$	$a_3$
$a_4$	$a_4$	$a_4$	$a_4$	$a_4$	$a_1$	$a_1$	$a_1$	$a_1$	$a_1$

**Figure 1: Two districts with a Nash equilibrium when considered separately, but none when considered together.**

of 18 voters, with preferences as depicted. Assume, for now, that parties can only choose to nominate their candidates in  $D_L$ .

Depending on whether party B, party C, or both nominate their candidate in the district, party A has a way to guarantee its candidate  $a_1$  to be elected. However, how to do so is not entirely obvious, as simply nominating  $a_1$  only will not work. If both  $b$  and  $c$  are nominated, party A must nominate all of its candidates. Candidate  $a_2$  then prevents candidate  $b$  from getting 7 votes, and candidate  $a_3$  withholds candidate  $c$  from getting 6 votes, making  $a_1$  the winner with only 5 votes. In other words, the role of  $a_2$  and  $a_3$  is that of splitting the votes of, respectively,  $b$  and  $c$  in favour of  $a_1$ . If, on the other hand, B nominates  $b$  but C does not nominate  $c$ , then party A must nominate  $a_1$  and  $a_2$ , but not  $a_3$ . In that way,  $a_1$  gets 8 votes against  $b$  getting 7 votes, and wins the election. By additionally nominating  $a_3$ , party A would split its own vote instead and candidate  $a_1$  would get no more than 5 votes. Similarly, if  $c$  is nominated, but  $b$  is not, party A must have  $a_1$  and  $a_3$  run, but not  $a_2$ . In conclusion, A has a strategy to win the district for each choice of the other parties, which implies there is at least one pure strategy Nash equilibrium.

Let us now look at the district in the right of Figure 1, call it  $D_R$ . This behaves symmetrically with respect to  $D_L$ , except that the roles of  $a_1$  and  $a_4$  are now inverted. Clearly, the equilibria of  $D_L$  carry over to  $D_R$ , with  $a_4$  replacing  $a_1$ . However, assume the two districts are now part of the same election and parties need to decide whom to nominate in each one of them. For example, A can decide to nominate  $a_4$  in  $D_R$  and nominate all others in  $D_L$ . This new scenario has no pure strategy Nash equilibrium. First of all notice that A cannot win both districts in equilibrium. When  $b$  and  $c$  are both in  $D_L$ , A wins both by nominating only  $a_4$  in  $D_R$  (adding other candidates would make A lose  $D_L$ ), but this causes  $b$  to deviate to  $D_R$ . When only  $c$  is in  $D_L$ , A needs  $a_1$  and  $a_3$  and not  $a_2$  there, but this makes  $c$  want to deviate to  $D_R$ , causing, in turn,  $b$  to deviate back to  $D_L$ . To face  $b$  only in  $D_L$ , A needs  $a_1$  and  $a_2$  but not  $a_3$ , which would however make  $b$  want to deviate to  $D_R$  and, in turn,  $c$  back to  $D_L$ . Similar arguments hold for  $b$  and  $c$ , both being in  $D_R$ . In light of the above, one can easily verify that all

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*profiles admit a profitable deviation and therefore are never a pure strategy Nash equilibrium.*

Strategic voting (see [17] for a comprehensive account) is one of the more active topics in multi-agent systems, in particular within the area of computational social choice. So far, the treatment of strategic behaviour has mainly focused on the voters’ side, allowing them to manipulate their preferences in order to elect a favourable candidate. A dual approach, closer in spirit to ours, is *strategic candidacy* [3, 8, 14], where candidates can decide to drop from elections they can never win in order for more preferred competitors to get an edge. Recent contributions, for instance [12], have explored strategic party nomination when elections are held in a Hotelling-Downs model, but an equilibrium analysis of district-based elections is surprisingly missing from the literature.

*Contribution.* In this paper we provide an equilibrium analysis of nominee selection in district-based elections, modelling voters as holding preferences over candidates and parties choosing which candidates to field in which district. We analyse various restrictions, such as voters following an underlying party preference (what we call party-oriented) or voters in each district being in agreement of which candidates from a party are closer to their needs (what we call community-oriented), and their repercussion in terms of equilibrium existence. Furthermore, we show that deciding the existence of pure Nash equilibria for these games is NP-complete if party size is bounded by a constant and  $\Sigma_2^P$ -complete for the general case. The proof of  $\Sigma_2^P$ -hardness proceeds by a reduction from  $\exists\exists!$ -3SAT, a variant of QSAT<sub>2</sub> which was shown to be  $\Sigma_2^P$ -complete by Marx [16].

Although we model voters holding an ideological position they do not change strategically, we see this work as a contribution to strategic voting [17], with participants trying to maximise their own electoral return.

The Nash equilibrium analysis of elections, in particular large ones such as national elections, is often met by the criticism that voters’ behaviour tends to be non-atomic, in the sense that strategising often results in no change of the overall outcome, also known as the *paradox of voting* or *Downs’ paradox* [7].

Our contribution gives, in our view, new vigour to the Nash Equilibrium analysis of large elections, where a small pool of parties can alter the choices of a large pool of voters by strategically selecting their candidates. While individual voters often act in highly incomplete information and limited computational resources, parties are structured entities that invest large amounts of resources in optimising their decision-making.

*Paper Structure.* In Section 2, we review and discuss related work. Section 3 presents the basic model setup and motivation. Section 4 presents equilibrium existence results focusing on restricted instances. Section 5 provides our core complexity results. We conclude in Section 6 with a discussion of our results.

## 2 RELATED WORK

Our framework relates to a number of research lines in computational social choice and algorithmic game theory:

*Strategic candidacy*, the decision of candidates not to run in an election, to see a more preferred candidate win [3, 8, 14]. Although

we restrict the analysis to parties that only care about the number of their elected candidates, in our model they can strategically place candidates in various districts, and strategically decide not to run in others, as in the scenario depicted in Example 1. This is also related to *election control*, the decision of external authorities to manipulate the elections by promoting (constructive control [1]) or hindering (destructive control [13]) certain candidates (see also [10]). In our framework parties can be seen as such authorities, who can influence elections without the need to resort to bribery [9].

*Primaries*, the internal party deliberations to put forward candidates for forthcoming elections [2], and strategic nominations [12], i.e., the selection of such candidates as a function of the other parties’ choices. In our framework parties compete over a number of districts, which reflects real-world elections systems and has repercussions on equilibrium existence and computation. Unlike [12], we allow for multiple elections occurring at the same time, and we do not restrict ourselves to specific preference structures.

*Colonel Blotto games* and their application to strategic decisions in elections [18] [15], specifically variants of the game with multifaceted resources, where the “type” of troops deployed matters [6]. Although in our framework parties strategically deploy resources, i.e., candidates, in territories, i.e., districts, the strength of the resources is determined by their relative rank compared to the ones deployed by the opposing parties. Because of the nature of voters’ preferences, the utility of parties is not monotonic, i.e., adding more candidates may harm a parties’ chances as it may end up splitting the vote. Recall from Example 1 that fielding more candidates may be useful to a party, as splitting the vote can have a positive strategic outcome, but this may also work against them.

*Distortion and misrepresentation.* The very nature of districts can create a mismatch between the majorities made by district winners and the overall proportional representation. In the 2019 General Elections in the UK, the Conservative Party won 56% of the seats, i.e., a solid overall majority, with a 44% of the votes. The field of computational social choice has looked at the divide between agents’ real preferences and the constraints imposed by electoral system, in terms of the distortion imposed by individual preferences or the misrepresentation when comparing them to voting outcomes [23], recently looking at these concepts in candidate selection inside political parties [2]. Although this paper is not directly concerned with these concepts, we offer a reality-resembling framework where they can naturally be applied.

*Team-based hedonic games.* Our work is also connected to the concept of stability in team-based hedonic games [26], because of the focus on equilibria resulting from coalitional choices, and tournament-based competitions such as the recent computational account of the Pokémon-Go territorial competitions with an underlying tournament structure [5]. We note in this respect that in the context of district-based elections, plurality is not reducible to a tournament solution and the winner in a district does not only depend on pairwise majority comparisons. As a matter of fact, in Example 1, both districts have a transitive majority relation, e.g.,  $b \succ_{maj} c \succ_{maj} a_1 \succ_{maj} a_2 \succ_{maj} a_3 \succ_{maj} a_4$  for  $D_L$ . Obviously, a district with all players having this relation as their preference relation, would have the same majority relation. Yet  $b$  would be elected whenever it were nominated in the district. By contrast,

who is elected in the district depends on the whole choice function induced by the plurality rule.

### 3 PRELIMINARIES

*Voters, Candidates, Districts and Assemblies.* Consider a set  $V = \{v_1, \dots, v_{|V|}\}$  of **voters**, partitioned into a set  $\mathbf{D} = \{D_1, \dots, D_{|\mathbf{D}|}\}$  of **districts** or **constituencies**, and a set  $C = \{c_1, \dots, c_{|C|}\}$  a set of **candidates**, partitioned into a set  $\mathbf{P} = \{P_1, \dots, P_{|\mathbf{P}|}\}$  of parties. We will at times use the auxiliary notion of **region**, understood as a subset of the set  $\mathbf{D}$  of districts, where we allow regions to overlap.

We assume elections to be simultaneously held in each district. Before the vote takes place, parties choose, independently and concurrently, which candidates to nominate in which district. Some parties may not have enough candidates to cover all district, and will need to choose among them. Other parties may have too many candidates and will need to choose who to leave out. After the parties have made their decision, voters in each district will choose among the candidates nominated in their district.

Formally, each district  $D$  is to elect  $\kappa$  representatives to take a seat in an assembly of representatives or parliament. For simplicity, we assume  $\kappa$  to be identical for all districts. We at times simplify this even further, and take  $\kappa$  to be 1, that is, every district elects no more than one representative in the assembly.

For a set  $X$ , we denote by  $\binom{X}{k}$  the set of subsets of  $X$  of size  $k$ . The set of **full assemblies** is thus given by  $\binom{C}{\kappa \cdot |\mathbf{D}|}$ . For technical convenience we also want to account for districts not being represented or for them being underrepresented, for instance, if fewer than  $\kappa$  candidates are running in a district, we define the set  $\mathbf{A}$  of **all possible assemblies**, the outcomes of our games, as  $\bigcup_{0 \leq k \leq \kappa \cdot |\mathbf{D}|} \binom{C}{k}$ . We will therefore assume that  $|C| \geq \kappa \cdot |\mathbf{D}|$ .

*Voters' Preferences.* We assume each voter  $i$  to have (**voter**)-**preferences** over the candidates, expressed as a total ordering  $\succsim_i$  over  $C$ . Similarly, each party  $P$  has preferences over the possible assemblies, also represented by a weak order  $\succsim_P$  over  $\mathbf{A}$ . For example, a party may prefer assemblies with a higher number of its elected candidates and is indifferent to the remaining composition.

The way candidates are chosen is as follows. In each district  $D$  each party nominates a number of their own candidates and a number of  $\kappa$  representatives are chosen among the candidates nominated in  $D$  on basis of the preferences of the voters in  $D$  and a given voting rule  $f$ .

Formally, a  $\kappa$ -**voting rule**  $f$  associates to every voter's profile  $\succsim = (\succsim_{v_1}, \dots, \succsim_{v_{|V|}})$ , every subset  $W$  of voters, and every subset  $K$  of candidates, a set of  $\kappa$  candidates from  $K$ , that is,  $f(\succsim, W, K) \in \binom{K}{\kappa}$ . In case  $|K| < \kappa$ , we set  $f(\succsim, W, K) = K$ . Clearly, if  $\kappa = 1$  and  $K \neq \emptyset$ , we have that  $f(\succsim, W, K)$  is a single candidate in  $K$ .

Although our framework allows for general voting rules, we will focus on the **plurality rule**, i.e., we elect a candidate in a district if they are ranked first by most voters in that district, assuming a fixed deterministic tie-breaking order over  $C$ .

*Strategies and Equilibria.* The issue we want to focus on is how the parties can manipulate the outcome of the election, that is, the composition the assembly elected, by strategically choosing which of their candidates are to run in which districts.

We formally define a (nomination) strategy for a party  $P$  as a **nomination function**  $v_P: \mathbf{D} \rightarrow 2^P$ . We require that a party can nominate a given candidate in at most one district, that is,  $D \neq D'$  implies  $v(D) \cap v(D') = \emptyset$ .

A **nomination profile** is then a profile  $v = (v_{P_1}, \dots, v_{P_{|\mathbf{P}|}})$  of nomination functions, one for each party. By  $v(D)$  we then denote the set of all candidates nominated by the parties in district  $D$ , that is,  $v(D) = \bigcup_{P \in \mathbf{P}} v_P(D)$ . Then,  $f(\succsim, D, v(D))$  is the set of candidates chosen to be represent district  $D$  in the assembly.

Given a set  $\mathbf{D} = \{D_1, \dots, D_{|\mathbf{D}|}\}$  of districts, a voter preference profile  $\succsim = (\succsim_{v_1}, \dots, \succsim_{v_{|V|}})$  and voting rule  $f$ , each nomination profile  $v = (v_{P_1}, \dots, v_{P_{|\mathbf{P}|}})$  now determines an assembly defined as

$$A(v) = \bigcup_{D \in \mathbf{D}} f(\succsim, D, v(D)).$$

A candidate  $c$  is said to be **elected under**  $v$  if  $c \in A(v)$ .

Finally, we assume each party to have (**party**)-**preferences** over the assemblies, represented by a reflexive, transitive, and connected relation  $\succsim_P$  over  $\mathbf{A}$ , with strict and indifferent parts  $>_P$  and  $\sim_P$ , respectively. A natural choice for party preferences, and the one adopted in this paper, is to assume that a party  $P$  prefers assembly  $A$  to assembly  $A'$  if the number of  $P$ 's members elected in  $A$  is larger than those elected in  $A'$ , that is, if

$$\sum_{D \in \mathbf{D}} |A_D \cap P| \geq \sum_{D \in \mathbf{D}} |A'_D \cap P|.$$

The parties, as players, together with their nomination strategies and preferences over assemblies, now define a strategic-form game, which we shall henceforth refer to as a **district-nomination game**, for which the usual game-theoretic solution concepts are defined. This in particular holds for (**pure**) **Nash equilibrium**, which then is a nomination profile  $v = (v_{P_1}, \dots, v_{P_{|\mathbf{P}|}})$  such that for all parties  $P$  and all strategies  $v'_P$ , we have

$$A(v) \succsim_P A(v_{-P}, v'_P),$$

where  $(v_{-P}, v'_P)$  is the profile  $v'' = (v''_{P_1}, \dots, v''_{P_{|\mathbf{P}|}})$  such that  $v''_P = v'_P$  and  $v''_Q = v_Q$  for all parties  $Q$  other than  $P$ . If a nomination profile  $v$  is a Nash equilibrium, we also say that  $v$  is **stable**.

### 4 EQUILIBRIA

Example 1 showed an instance of district-based elections where individual districts have a number of equilibria if taken in isolation, but putting them together does not guarantee a stable outcome. In this section, we want to understand whether equilibria exist as we alter the game structure, for example the number of districts or the preference structure.

Observe first that the presence of districts is not necessarily the reason why equilibria do not exist. In fact, if there are at least as many districts as there are candidates, equilibria are always guaranteed to exist, as each candidate can be nominated in a separate district, constructing an equilibrium. When parties compete for scarce districts, unstable elections are in fact ubiquitous, even in very simple setups. Figure 2 presents an instance of an unstable election with two parties and one district.

**PROPOSITION 1.** *There is a two-party one-district game with no pure Nash equilibria.*

	$\emptyset$	$b_1$	$b_2$	$b_1 b_2$
$\emptyset$	– 0, 0, 0, 0	$b_1$ 0, 0, 42, 0	$b_2$ 0, 0, 0, 42	$b_1 b_2$ 0, 0, 29, 13
$a_1$	$a_1$ 42, 0, 0, 0	$a_1$ 22, 20, 0, 0	$b_2$ 16, 0, 0, 26	$b_2$ 12, 0, 20, 10
$a_2$	$a_2$ 0, 42, 0, 0	$b_1$ 0, 12, 30, 0	$a_2$ 0, 25, 0, 17	$b_1$ 0, 12, 20, 10
$a_1 a_2$	$a_1$ 33, 9, 0, 0	$a_1$ 22, 9, 11, 0	$b_2$ 16, 9, 0, 17	$a_1$ 12, 9, 11, 10

**Figure 2: On the left the preference rankings of the voters. On the right the normal-form game resulting from the nomination strategies of parties  $A = \{a_1, a_2\}$  and  $B = \{b_1, b_2\}$ . The entries in the matrix specify the winner and the vote breakdown for each candidate, in lexicographic order. The game obtained has no pure Nash equilibrium.**

The fact that Nash equilibria are not guaranteed to exist even in the most basic scenarios motivates the computational questions we ask in Section 5. However, there are restrictions in terms of district size and voters’ composition, where equilibria do instead exist and warrant consideration. This section is devoted to a few basic observations on this, complementing the complexity analysis.

To start, we say that a preference profile  $\succsim = (\succsim_{v_1}, \dots, \succsim_{v_{|V|}})$  is **community-oriented** if for each two candidates  $c_i, c_j$  belonging to the same party and each two voters  $v_i, v_j$  belonging to the same district, we have that  $c_i \succsim_{v_i} c_j$  iff  $c_i \succsim_{v_j} c_j$ . With community-oriented preferences, voters can diverge on party choice, but, within each party, they agree on the candidates that best represent them. The idea is that some candidates are objectively closer to a district’s problems than others, e.g., they come from the same region, and voters agree on this.

Dually, the following restriction looks at parties first. We say that a preference profile  $\succsim = (\succsim_{v_1}, \dots, \succsim_{v_{|V|}})$  is **party-oriented** if for all candidates  $c_i, c_j, c_k$  with  $c_i, c_j \in P_i$  and  $c_k \in P_j \neq P_i$ , and each voter  $v_i$ , we have that  $c_i \succsim_{v_i} c_k$  implies that  $c_j \succsim_{v_i} c_k$ . In other words, party-oriented preferences are such that voters have a strict preference over parties, and they rank candidates from preferred parties always higher than those from less preferred ones. However, even if sharing the same preference over the parties, they may disagree on how to rank candidates within a party.

With party-oriented preferences, the party that is most often ranked on top in one district can always guarantee a candidate to be elected by simply fielding their strongest member according to the tie-breaking order, and this follows.

**PROPOSITION 2.** *With one district and party-oriented preferences, there is always a Nash equilibrium.*

This is also true for community-oriented preferences, as voters who prefer the same party also prefer the same candidate.

**PROPOSITION 3.** *With one district and community-oriented preferences, there is always a Nash equilibrium.*

When two parties compete, differences begin to arise in terms of stability. Say that a district is **oddly populated**, if the number of voters in that district is odd, **evenly populated**, otherwise.

**PROPOSITION 4.** *Let each district be oddly populated. Then two-party elections with party-oriented preferences always have a Nash equilibrium.*

**PROOF.** Let  $P, P'$  be two parties. Now call a district  $D$  a *safe seat* for party  $P$  if every single-candidate nomination by party  $P$  will get that candidate elected in  $D$ , no matter the response of  $P'$ . Observe that, with two-party elections, we have that each district with an odd number of voters is either a safe seat for party  $P$  or it is for party  $P'$ . Let now each party make a single-candidate nomination, until they exhaust their own safe seats or their own members. If all safe seats are occupied by some candidate, then we have a Nash equilibrium, as no party can profitably deviate by occupying the opposing party safe seat or adding more nominees in their own. Without loss of generality, assume that a safe seat by  $P'$  is unoccupied, instead. Let now  $P$  occupy it by one of their remaining candidates and proceed for each such seat until their remaining candidates are exhausted. As  $P'$  would have occupied the seat at the earlier stage if enough of their candidates were available, this is also a Nash equilibrium profile.  $\square$

Evenly populated districts allow for instability.

**PROPOSITION 5.** *There are two-party elections with party-oriented preferences and no Nash equilibrium.*

**PROOF.** Consider parties  $A = \{a_1, a_2\}$  and  $B = \{b\}$  with an underlying tie-breaking order of  $a_1 > b > a_2$  and two districts,  $D_1$  and  $D_2$ .  $D_1$  is populated by voter  $v_1$ , with preference relation  $a_1 > a_2 > b_1$ , and voter  $v_2$ , with preference relation  $b_1 > a_1 > a_2$ .  $D_2$  is populated by voter  $v_3$ , with the same preference as  $v_1$ , and voter  $v_4$ , with the same preference as  $v_2$ . We can observe that each nomination profile admits a profitable deviation. Whenever  $a_1$  is nominated in a district, say  $D$ ,  $B$  is better off nominating  $b_1$  in  $D'$ . But then  $A$  can win both district by simply following  $b_1$  in  $D'$  and leaving  $a_2$  in  $D$  instead.  $\square$

Using a similar idea we can show a broader result for community-oriented preferences.

**PROPOSITION 6.** *There are two-party elections with community-oriented preferences and no Nash equilibrium, independently on the number of oddly populated districts.*

**PROOF.** Consider  $A = \{a_1, a_2\}$ ,  $B = \{b\}$  and an underlying community order of  $a_1 > b > a_2$  in two districts  $D_1$  and  $D_2$ , each made by voters with preference order  $a_1 > b > a_2$ . This, notice, is independently on the number of oddly populated districts. Now, candidate  $b$  will always want to deviate to the district where  $a_1$  is not. But then  $A$  can field  $a_1$  there, and move  $a_2$  to the district where  $b$  is not, creating a cycle.  $\square$

As soon as the number of parties grows to three and there are at least two districts, Nash equilibria cease to be guaranteed, even under party-oriented preferences.

**PROPOSITION 7.** *There is a three-party two-district game where voters have party-oriented preferences and no Nash equilibrium.*

PROOF. Let parties  $A = \{a\}$ ,  $B = \{b\}$  and  $C = \{c\}$  compete for districts  $D_1$  and  $D_2$ .

Let each  $D_i$  be such that  $a$  beats  $b$ ,  $b$  beats  $c$  and  $c$  beats  $a$  in pairwise majority comparison, building a Condorcet cycle. Now, if not all candidates are present together in the same district, there will be a party that does not win either district. But then this party can successfully change their strategy and run in the other district, winning it. Likewise, if all parties run in the same district, two parties have a profitable deviation towards the other one. No party is better off when not running. So there is no Nash equilibrium.  $\square$

Although stability can be guaranteed in restricted instances, equilibria do not generally exist. This makes equilibrium computation an important challenge, which we tackle next.

## 5 COMPLEXITY

In this section, we explore the computational complexity of deciding whether a given district nomination game has a Nash equilibrium. Let us recall that we assume  $\kappa = 1$  and plurality as voting rule. We also assume that the parties are solely interested in getting as many of their candidates elected in the assembly. A district nomination game is thus fully determined by the voters' preferences over the candidates, the partition of the voters in districts, and the partition of the candidates in parties. Its size can be taken to be  $|V| \cdot |C|$ .

### DISTRICT CANDIDACY

*Given:* District nomination game  $G$  with set  $V$  of voters, set  $C$  of candidates, set  $D$  of districts, set  $P$  of parties

*Problem:* Does  $G$  allow for a pure Nash equilibrium?

We first prove that DISTRICT CANDIDACY is NP-hard by a reduction from the satisfiability problem 3SAT. We see that this reduction involves the construction of a district nomination game wherein all parties are singletons. For this special class, it is not hard to see that DISTRICT CANDIDACY is also contained in NP, giving us an NP-completeness result. For general district nomination games, we subsequently find that DISTRICT CANDIDACY is  $\Sigma_2^P$ -complete.

The proof of  $\Sigma_2^P$ -hardness proceeds by a reduction from  $\exists\exists!$ -3SAT, a variant of QSAT<sub>2</sub> which was shown to be  $\Sigma_2^P$ -complete by Marx [16] [also see 25]. The instances of  $\exists\exists!$ -3SAT are of the form  $\exists X\exists!Y\varphi$ , where  $\varphi$  is a propositional formula over  $\Phi$ , and  $X$  and  $Y$  are subsets of propositional variables partitioning  $\Phi$ . Then,  $\exists\exists!$ -3SAT is the decision problem whether there is an assignment of truth values to the variables in  $X$  such that there is *exactly one* assignment of truth values to the variables in  $Y$  satisfying a given Boolean formula  $\varphi$ . Importantly, for  $\exists\exists!$ -3SAT to remain  $\Sigma_2^P$ -hard, one may restrict attention to formulas  $\varphi$  in 3CNF form. This is similar to 3SAT, but different from the canonical  $\Sigma_2^P$ -complete problem QSAT<sub>2</sub>, the set of true formulas of the form  $\exists X\forall Y\varphi$ , which only remains  $\Sigma_2^P$ -hard if  $\varphi$  is in 3DNF form. Moreover, it enables us to show  $\Sigma_2^P$ -hardness of DISTRICT CANDIDACY by extending the 3-SAT construction that demonstrated its NP-hardness.

### 5.1 NP-hardness

Let an instance  $\varphi$  of 3SAT be given by a set  $K = \{K_1, \dots, K_{|K|}\}$  of pairwise distinct clauses, where each clause  $K$  in  $K$  is given by a set  $\{\ell'_K, \ell''_K, \ell'''_K\}$  of three literals over a set of propositional

3	2	2
$a_p$	$c_1$	$b_1$
$b_1$	$\vdots$	$\vdots$
$\vdots$	$c_{\hat{k}_\ell}$	$b_{k_\ell}$
$b_{k_\ell}$	$a_p$	$c_1$
$c_1$	$b_1$	$\vdots$
$\vdots$	$\vdots$	$c_{\hat{k}_\ell}$
$c_{\hat{k}_\ell}$	$b_{k_\ell}$	$a_p$
$e_1$	$e_1$	$e_1$
$e_2$	$e_2$	$e_2$

**Figure 3: Voter preferences over local candidates in district  $D_{\ell,i}$  with  $K_\ell = \{K_1, \dots, K_{k_\ell}\}$ . We write  $b_i$  for  $b_{K_i}$ , and  $c_i$  and  $e_i$  for  $c_{\ell,i}$  and  $e_{\ell,i}$ , respectively. Only preferences over local candidates are listed; preferences over non-local candidates are understood to be ordered in arbitrary order at the bottom of the preference lists.**

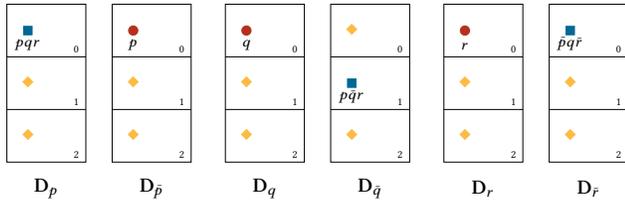
variables  $\Phi = \{p_1, \dots, p_{|\Phi|}\}$ . Thus, every literal is of the form  $p$  or  $\neg p$  from some  $p \in \Phi$ . We have  $\bar{\ell} = \neg p$  if  $\ell = p$ , and  $\bar{\ell} = p$  if  $\ell = \neg p$ . We generally also write  $\bar{p}$  for  $\neg p$  and omit parentheses and commas in clauses, writing, for instance,  $p\bar{q}\bar{r}$  for the clause  $\{p, \neg q, \neg r\}$ . For each literal  $\ell$ , we let  $K_\ell = \{K \in K: \ell \in K\}$ , that is,  $K_\ell$  is the set of all clauses in  $K$  that contain  $\ell$ . We set  $k_\ell = |K_\ell|$  and  $\hat{k}_\ell = \max(|K_\ell|, |K_{\bar{\ell}}|)$ .

Let  $\varphi$  be a 3CNF given by a set  $K$  of clauses over  $\Phi$ . We construct a district nomination game  $G_\varphi$  with a set of  $O(2|\Phi| + 3|K|)$  districts, each to be populated with seven voters below. For every literal  $\ell$ , we let  $D_\ell = \{D_{\ell,0}, \dots, D_{\ell,\hat{k}_\ell}\}$  be a set of  $\hat{k}_\ell + 1$  districts, which we will also refer to as a the **literal region** for  $\ell$ . The set  $D$  of all districts is then given by the union of all literal regions, that is,  $D = D_{\ell_1} \cup \dots \cup D_{\ell_{2|\Phi|}}$ , where  $\{\ell_1, \dots, \ell_{2|\Phi|}\}$  is the set of all literals over  $\Phi$ . The literal regions are pairwise disjoint and, thus, partition the set  $D$  of all districts.

As **candidates** we have, for each variable,  $p \in \Phi$ , one **variable candidate**  $a_p$ , for each clause  $K \in K$ , one **clause candidate**  $b_K$ , for each literal  $\ell$ , a total of  $\hat{k}_\ell$  **literal candidates**  $c_{\ell,1}, \dots, c_{\ell,\hat{k}_\ell}$ , and for each literal  $\ell$  two **“extra” candidates**  $e_{\ell,1}$  and  $e_{\ell,2}$ . Observe that we thus have  $O(5|\Phi| + 4|K|)$  candidates in total. As the **parties** of the district nomination game  $G_\varphi$ , we take the singleton coalitions, containing exactly one candidate. Thus, a party  $\{c\}$  strictly prefers assembly  $A$  to assembly  $A'$ , if  $c$  is elected as a representative in  $A$  but not in  $A'$ . Otherwise party  $\{c\}$  is indifferent. For notational convenience, we will generally identify singleton parties with their one candidate and simply write  $c$  for  $\{c\}$ . Moreover, with some abuse of notation, we also write  $v_c = D$  if  $v_c(D) = \{c\}$  (and, hence,  $v_c(D') = \emptyset$ , for all districts  $D' \neq D$ ).

We associate each candidate  $c$  with a region  $D_c \subseteq D$ . For each variable  $p$  in  $\Phi$ , each clause  $K = \{\ell', \ell'', \ell'''\}$ , and all literals  $\ell$ , we define, for  $1 \leq i \leq \hat{k}_\ell$  and  $1 \leq j \leq 2$ :

$$\begin{aligned} D_{a_p} &= D_p \cup D_{\bar{p}} & D_{b_K} &= D_{\ell'} \cup D_{\ell''} \cup D_{\ell'''} \\ D_{c_{\ell,i}} &= D_\ell & D_{e_{\ell,j}} &= D_\ell \end{aligned}$$



**Figure 4:** NP-construction for 3CNF  $\varphi$  with clauses  $K = \{pqr, p\bar{q}\bar{r}, \bar{p}q\bar{r}\}$ , depicting a stable nomination profile.

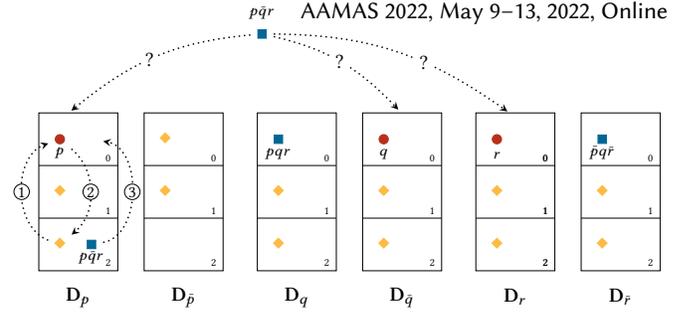
We say that candidate  $c$  is **local** to the districts in  $D_c$  and **non-local** to all other districts. We also say that  $c$  is local to literal region  $D_\ell$ , if  $D_\ell \subseteq D_c$ . We will later see that candidates can only hope to be elected in districts/regions where they are local.

We now populate the districts with **voters**. Let  $\ell$  be a literal. Then, for every  $0 \leq i \leq \hat{k}_\ell$ , the district  $D_{\ell,i}$  has seven voters whose preferences are depicted in Figure 3. The important thing to observe is that, in each district  $D_{\ell,i}$  with  $\ell$  a literal over  $p$ , there is a majority cycle between variable candidate  $a_p$ , each clause candidate  $b_K$  with  $K \in K_\ell$ , and each literal  $c_{\ell,i}$ . That is, a majority of voters prefers  $a_p$  to any  $b_K$  for  $K \in K_\ell$ , another majority of voters prefers each  $b_K$  to any  $c_{\ell,j}$ , and there is a third majority of voters preferring each  $c_{\ell,j}$  to  $a_p$ . If in some district  $D_{\ell,i} \in D_\ell$ , the local variable candidate  $a_p$ , a local clause candidate  $b_K$ , and a literal candidate  $c_{\ell,i}$  are all nominated, then  $a_p$  will win the plurality vote.

To illustrate our construction, consider the 3CNF  $\varphi$  with clauses  $K = \{pqr, p\bar{q}\bar{r}, \bar{p}q\bar{r}\}$ . For the literals, then  $K_p = \{pqr, p\bar{q}\bar{r}\}$  and  $K_{\bar{q}} = \{p\bar{q}\bar{r}\}$ . Each of the six literal regions has three districts, for instance,  $D_p = \{D_{p,0}, D_{p,1}, D_{p,2}\}$  and  $D_{\bar{r}} = \{D_{\bar{r},0}, D_{\bar{r},1}, D_{\bar{r},2}\}$ . See Figure 4 for the setup of the districts and literal regions. There, local variable candidates  $a_p$  are depicted by red circles ( $\bullet_p$ ), local clause candidates  $b_K$  by blue squares ( $\blacksquare_K$ ), and local literal candidates  $c_{\ell,i}$  by yellow diamonds ( $\blacklozenge$ ) in the districts in which they are nominated, that is, Figure 4 depicts a nomination profile. Thus, variable candidate  $a_p$  is nominated in  $D_{\bar{p},0}$ , clause candidate  $b_{p\bar{q}\bar{r}}$  in  $D_{\bar{q},1}$ , and literal candidate  $c_{q,1}$  in  $D_{q,1}$ . Extra and non-local candidates are omitted, as the former ensure that the latter are elected in equilibrium, but otherwise play no role in the equilibrium analysis.

Nominations of the three variable candidates,  $a_p$ ,  $a_q$ , and  $a_r$ , determine a truth-value assignment for the variables  $p$ ,  $q$ , and  $r$ , provided they are nominated in districts they are local to. If  $a_x$  is nominated in a district in  $D_x$ , variable  $x$  is set to false, if  $a_x$  is nominated in a district in  $D_{\bar{x}}$ , variable  $x$  is set to true. Thus, in our example in Figure 4, variable  $p$  is to true and  $q$  and  $r$  to false. Also, observe that this assignment *satisfies*  $\varphi$ . What is more, this nomination profile is *stable*, as all local variable, clause, and literal candidates will be elected and, hence, do not want to deviate.

Now consider Figure 5, which depicts another nomination profile in the same game, determining the assignment that sets  $p$  and  $r$  to false, and  $q$  to true and does not satisfy  $\varphi$ . The nomination profile is not stable either. To see this, consider clause candidate  $b_{p\bar{q}\bar{r}}$ , who can only hope to be elected (under Nash equilibrium) if nominated in  $D_p$ ,  $D_{\bar{q}}$ , or  $D_r$ . If so, however,  $b_{p\bar{q}\bar{r}}$  is bound to get entangled in a nomination cycle. For instance, if  $b_{p\bar{q}\bar{r}}$  is nominated in  $D_{p,2}$  in  $D_p$ , the literal candidate nominated in  $D_{p,2}$  would deviate



**Figure 5:** NP-construction for 3CNF  $\varphi$  with clauses  $K = \{pqr, p\bar{q}\bar{r}, \bar{p}q\bar{r}\}$ , depicting a unstable nomination profile. Clause candidate  $b_{p\bar{q}\bar{r}}$  cannot be nominated in any of its local literal regions without getting into a cycle.

to  $D_{p,0}$  (①), whereupon, variable candidate would move to  $D_{p,2}$  (②), inciting  $b_{p\bar{q}\bar{r}}$  to move to  $D_{p,0}$  (③), and so on.

This argument can be generalised and we find that a nomination profile in  $G_\varphi$  is stable if and only if  $\varphi$  is satisfiable. Our argument for NP-hardness of DISTRICT CANDIDACY relies on four observations.

- (1) In all districts, all voters prefer local candidate to non-local candidates. This entails that, in equilibrium, candidates can only be elected if nominated in their own region.
- (2) Variable and clause candidates always have a best response that gets them elected. In particular, for  $\ell$  a literal on  $p$ , we have  $|D_\ell| > \hat{k}_\ell$ . Hence, there will always be a district in  $D_\ell$  not occupied by  $c_{\ell,i}$  where  $a_p$  can be elected. All variable and all clause candidates will thus be elected in equilibrium.

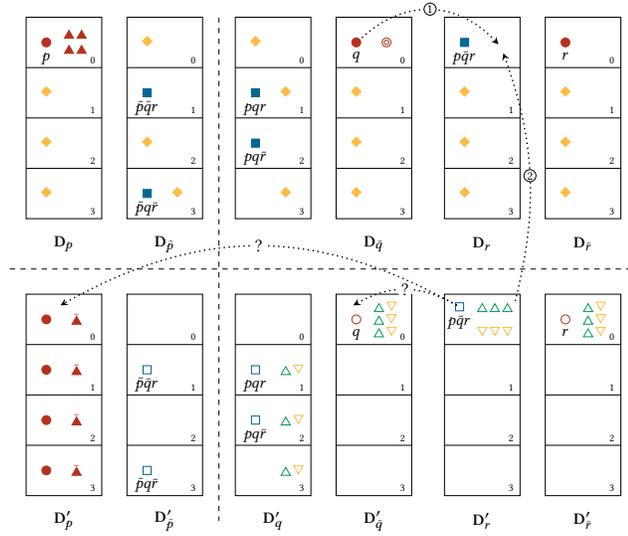
On this basis we define a **pre-equilibrium** as a nomination profile wherein all variable and clause candidates are nominated in districts where they are local, that is, if  $v_{a_p} \in D_{a_p}$  and  $v_{b_K} \in D_{b_K}$ . A pre-equilibrium  $v$  defines an assignment  $\alpha$ , such that, if under  $v$  variable candidate  $a_p$  is nominated in  $D_p$ , then  $\alpha$  sets  $p$  sets to false, and, if  $P$  nominates  $a_p$  in  $D_{\bar{p}}$ , then  $\alpha$  sets  $p$  to true. This mapping between nominations of variable candidates and assignments is one-to-one and onto. We furthermore say that a nomination profile  $v$  **separates** candidates  $c$  and  $c'$  if (i) they are nominated under  $v$ , and (ii) they are nominated in different literal regions. We find that a pre-equilibrium  $v$  separates each variable candidate from all clause candidates if and only if the valuation induced by  $v$  satisfies  $\varphi$ . On this basis, we obtain the next two observations.

- (3) The game  $G_\varphi$  has a Nash equilibrium if and only if it allows for a pre-equilibrium that separates each variable candidate  $a_p$  from all clause candidates  $b_K$ . The key to this observation is that if  $a_p$  and  $b_K$  are not separated, they are bound to get entangled in a majority cycle with the literal candidates of the literal region in question.
- (4) A 3CNF  $\varphi$  is satisfiable if and only if  $G_\varphi$  allows for a pre-equilibrium that separates each variable candidate  $a_p$  from all clause candidates  $b_K$ .

As an immediate consequence of observations 3 and 4, we can now state our first main result of this section.

**THEOREM 1.** *DISTRICT CANDIDACY is NP-hard.*

If the parties—as in the construction of  $G_\varphi$ —all have exactly one candidate, then they have each exactly  $|D| + 1$  strategies at their disposal. Thus, one can ‘guess’ a nomination profile and check in polynomial time whether no party has an incentive to deviate to



**Figure 6:**  $\Sigma_2^P$ -construction for instance  $\exists\{p\}\exists!\{q,r\}\varphi$  where  $\varphi$  is given by  $\{pqr, pq\bar{r}, p\bar{q}r, \bar{p}q\bar{r}, \bar{p}\bar{q}r\}$ . Elected candidates are depicted leftmost in each district.

one of its  $|D| + 1$  strategies. This observation extends to the case where the size  $|P|$  of each party  $P$  is fixed at a constant and the size of its strategy space  $(|D| + 1)^{|P|}$  bounded by a polynomial.

**COROLLARY 1.** *If all parties have a constant number of candidates, DISTRICT CANDIDACY is NP-complete. This, in particular holds, for the case where each party has exactly one candidate.*

## 5.2 $\Sigma_2^P$ -Completeness

Corollary 1 still leaves open the upper bound of the complexity of DISTRICT CANDIDACY for the general case, where party size is not bounded by a constant. We find that this problem is  $\Sigma_2^P$ -complete. To demonstrate this, we first consider EQUILIBRIUM VERIFICATION, the decision problem whether a given nomination profile  $\nu$  is a pure Nash equilibrium in a given district nomination game. This problem is in coNP: given a nomination profile  $\nu$ , one can non-deterministically guess a party  $P$  along with one of its strategies  $\nu'_P$ , and check if  $P$  wants to deviate to  $\nu'_P$ . Recalling that  $\Sigma_2^P = \text{NP}^{\text{coNP}}$ , membership of DISTRICT CANDIDACY in  $\Sigma_2^P$  follows almost immediately: one can non-deterministically guess a nomination profile and consult the coNP-oracle to check whether it is stable.

**LEMMA 1.** *EQUILIBRIUM VERIFICATION is in coNP. As a consequence, DISTRICT CANDIDACY is in  $\Sigma_2^P$ .*

To prove  $\Sigma_2^P$ -hardness of DISTRICT CANDIDACY, we extend the construction we used to prove its NP-hardness. Let  $\exists X\exists!Y\varphi$  an instance of  $\exists\exists!3\text{-SAT}$ ; we construct a district nomination game  $G_{\exists X\exists!Y\varphi}$  that has a Nash equilibrium if and only if  $\exists X\exists!Y\varphi$  holds. For technical purposes, we will assume that the set  $\mathbf{K}$  of clauses of  $\varphi$  contains  $Y = \{\{y, \bar{y}\} : y \in Y\}$ . We can make this assumption without loss of generality, as  $\exists X\exists!Y\varphi$  holds if and only if  $\exists X\exists!Y\varphi'$ , when  $\varphi$  is given by  $\mathbf{K}$  and  $\varphi'$  by  $\mathbf{K} \cup Y$ .

We introduce two literal regions  $\mathbf{D}_\ell = \{D_{\ell,0}, \dots, D_{\ell,\hat{k}_\ell}\}$  and  $\mathbf{D}'_\ell = \{D'_{\ell,0}, \dots, D'_{\ell,\hat{k}_\ell}\}$  for each literal  $\ell$  over  $\Phi$ . For an illustration,

see Figure 6. In this context, we say that a nomination profile  $\nu$  **separates** candidates  $c$  and  $c'$ , if both (i)  $c$  and  $c'$  are nominated under  $\nu$ , and (ii)  $\nu_c \in \mathbf{D}_\ell \cup \mathbf{D}'_\ell$  and  $\nu_{c'} \in \mathbf{D}_{\ell'} \cup \mathbf{D}'_{\ell'}$  imply  $\ell \neq \ell'$ .

We populate these districts with candidates who belong to five parties  $P, Q, B, C$ , and  $E$ . **Party P** consists of one *variable candidate*  $a_p$  for each variable  $p \in \Phi$ , a total of  $\hat{k}_x + 1$  *variable candidates* denoted by  $a'_{x,0}, \dots, a'_{x,\hat{k}_x}$  for each variable  $x \in X$ , and  $\hat{k}_\ell + 1$  *literal candidates*  $d_{\ell,0}, \dots, d_{\ell,\hat{k}_\ell}$  for each literal  $\ell$  over  $X$ . **Party Q** consists of one *variable candidate*  $a'_y$ , for each variable  $y \in Y$ , one *clause candidate*  $b'_K$  for each clause  $K \in \mathbf{K}$ , and  $2 \cdot \hat{k}_\ell$  *literal candidates*  $d'_{\ell,1}, \dots, d'_{\ell,\hat{k}_\ell}$ , and  $d''_{\ell,1}, \dots, d''_{\ell,\hat{k}_\ell}$  for each literal  $\ell$  on  $Y$ . In addition, party  $Q$  also has one **spoiler candidate**  $a''$ . **Party B** has one *clause candidate*  $b_K$  for each clause  $K \in \mathbf{K}$ . For each literal  $\ell$  on  $\Phi$ , **Party C** consists of  $\hat{k}_\ell$  *literal candidates* denoted by  $c_{\ell,1}, \dots, c_{\ell,\hat{k}_\ell}$ . Finally, **Party E** consists of  $\hat{k}_\ell + 3$  “extra” candidates  $e_{\ell,1}, e_{\ell,2}, e'_{\ell,1}, \dots, e'_{\ell,\hat{k}_\ell+2}$  for each literal  $\ell$  on  $\Phi$ .

We now define for each candidate  $c$  a region  $\mathbf{D}_c$  where  $c$  is considered to be local. For each variable  $p$  in  $\Phi$ , each clause  $K = \{\ell', \ell'', \ell'''\}$ , all literals  $\ell, 1 \leq i \leq r_\ell$ , and  $1 \leq j \leq 2$ , we have:

$$\begin{aligned} \mathbf{D}_{a_p} &= \mathbf{D}_p \cup \mathbf{D}_{\bar{p}} & \mathbf{D}_{b_K} &= \mathbf{D}_{\ell'} \cup \mathbf{D}_{\ell''} \cup \mathbf{D}_{\ell'''} & \mathbf{D}_{d'_{\ell,i}} &= \mathbf{D}'_\ell \\ \mathbf{D}_{a'_{x,i}} &= \mathbf{D}'_x \cup \mathbf{D}'_{\bar{x}} & \mathbf{D}_{a'_y} &= \mathbf{D}'_y \cup \mathbf{D}'_{\bar{y}} & \mathbf{D}_{d''_{\ell,i}} &= \mathbf{D}_\ell \cup \mathbf{D}'_\ell \\ \mathbf{D}_{d_{\ell,i}} &= \mathbf{D}_\ell \cup \mathbf{D}'_\ell & \mathbf{D}_{a''} &= \bigcup_{y \in Y} (\mathbf{D}_y \cup \mathbf{D}_{\bar{y}}) & \mathbf{D}_{e_{\ell,j}} &= \mathbf{D}_\ell \\ \mathbf{D}_{c_{\ell,i}} &= \mathbf{D}_\ell & \mathbf{D}_{b'_K} &= \mathbf{D}'_{\ell'} \cup \mathbf{D}'_{\ell''} \cup \mathbf{D}'_{\ell'''} & \mathbf{D}_{e'_{\ell,i}} &= \mathbf{D}'_\ell \end{aligned}$$

Appropriate voter preferences can now be defined. To do so, we partition the districts into four regions:  $\mathbf{D}_X = \bigcup_{x \in X} (\mathbf{D}_x \cup \mathbf{D}_{\bar{x}})$ ,  $\mathbf{D}_Y = \bigcup_{y \in Y} (\mathbf{D}_y \cup \mathbf{D}_{\bar{y}})$ ,  $\mathbf{D}'_X = \bigcup_{x \in X} (\mathbf{D}'_x \cup \mathbf{D}'_{\bar{x}})$ , and  $\mathbf{D}'_Y = \bigcup_{y \in Y} (\mathbf{D}'_y \cup \mathbf{D}'_{\bar{y}})$ . If two districts both belong to one of these regions, they have very similar electorates.<sup>1</sup>

On  $\mathbf{D}_X$  and  $\mathbf{D}_Y$ , the game  $G_{\exists X\exists!Y\varphi}$  is reminiscent of  $G_\varphi$ . Intuitively, satisfiability of  $\varphi$  is checked there. If that is the case, party  $Q$  then tries to find an alternative assignment for  $Y$  in  $\mathbf{D}'_X$  and  $\mathbf{D}'_Y$  so as to falsify  $\exists X\exists!Y\varphi$ .

The definition of the voter preferences ensure that observations (1) and (2) regarding  $G_\varphi$  extend in adapted form to  $G_{\exists X\exists!Y\varphi}$ : in equilibrium candidates are elected in districts where they are local, and parties  $P$  and  $B$  always have a best-response in which all of their variable and clause candidates are elected. Moreover,  $P$ 's and  $Q$ 's literal candidates will never be elected in equilibrium. Their role is rather to split the vote for their party's variable and clause candidates. Thus, for  $\ell$  a literal over  $p$ , party  $P$ 's literal candidates  $d_{\ell,i}$  guarantee that  $a_x$  is nominated in  $\mathbf{D}_\ell$  if and only if all of  $P$ 's variable candidates  $a'_{x,j}$  are elected in  $\mathbf{D}'_\ell$ . This forces  $Q$ 's clause candidates to be nominated in  $\mathbf{D}'_\ell$  if they are to be elected. Similarly,  $Q$ 's literal candidates  $d'_{\ell,i}$  and  $d''_{\ell,j}$  ensure that  $Q$ 's variable and clause candidates will only be *all* elected if they are separated. Furthermore, all of  $Q$ 's literal candidates  $d''_{\ell,i}$  are needed to get  $Q$ 's spoiler candidate elected in a district  $D_{\ell,j}$ . Finally, the inclusion of  $Y$  in  $\mathbf{K}$  ensures that for each variable  $\ell$  at least one clause candidate is nominated in some district in  $\mathbf{D}_\ell \cup \mathbf{D}_{\bar{\ell}}$ .

We now argue that  $\exists X\exists!Y\varphi$  holds if and only if  $G_{\exists X\exists!Y\varphi}$  has a Nash equilibrium, which suffices for a proof of  $\Sigma_2^P$ -hardness of

<sup>1</sup>The voters' preferences in each of these regions are depicted in Figures A1 through A4 in the appendix at <https://www.dropbox.com/s/h71pp10yhrs8c/h/appendix.pdf?dl=0>.

**DISTRICT CANDIDACY.** The key observation is again that there is a one-one correspondence between party  $P$ 's and party  $Q$ 's nominations of variable candidates and truth-value assignments. Thus,  $a_p$  is nominated in  $D_{\bar{p}}$  (in  $D_p$ ) if and only if  $p$  is set to true (to false). Similarly,  $a'_y$  is nominated in  $D'_y$  (in  $D'_y$ ) if and only if  $y$  is set to true (to false). Note that the nominations of  $P$ 's candidates and those of  $Q$ 's candidates for a single nomination profile may correspond to different assignments. A similar equivalence holds between nominations that separate  $Q$ 's clause candidates  $b'_K$  from  $P$ 's variable candidates  $a_x$  with  $x \in X$  and its own variable candidates  $a_y$  on the one hand and assignments satisfying  $\varphi$  on the other.

First assume that  $\exists X \exists Y \varphi$  holds, that is, there is an assignment  $\alpha$  satisfying  $\varphi$  such that the only assignment that coincides with  $\alpha$  on  $X$  and still satisfies  $\varphi$  is  $\alpha$  itself. Then, it is possible to construct a stable nomination profile such that the nominations of  $P$ 's and  $Q$ 's variable candidates both correspond to  $\alpha$ . As  $\varphi$  is satisfiable,  $B$ 's and  $Q$ 's clause candidates will be separated from  $P$ 's and  $Q$ 's variable candidates. Accordingly, all variable and clause candidates are elected. This is illustrated in Figure 6, which concerns the instance  $\exists\{p\}\exists!\{q, r\}\varphi$  where  $\varphi$  be given by  $\{pqr, pq\bar{r}, p\bar{q}r, \bar{p}q\bar{r}, \bar{p}\bar{q}r\}$ .<sup>2</sup> Note that there is only one assignment that sets  $p$  to false and satisfies  $\varphi$ , namely, the assignment  $\alpha_{\bar{p}qr}$  that additionally sets  $q$  and  $r$  to true. Hence,  $\exists\{p\}\exists!\{q, r\}\varphi$  holds. The nominations of party  $P$ 's variable candidates  $a_p$  ( $\bullet_p$ ) correspond to the assignment  $\alpha_{\bar{p}qr}$ . Moreover,  $P$ 's variable candidates  $a_p$  are separated from  $B$ 's clause candidates  $b_K$  ( $\blacksquare_K$ ) reflecting that  $\alpha_{\bar{p}qr}$  satisfies  $\varphi$ . Thus,  $P$  and  $B$  will get all their non-literal candidates elected, and do not want to deviate. Note we may assume that party  $C$ 's nominations of its literal candidates ( $\blacklozenge$ ) constitute a best-response, and that party  $E$  (not depicted) does not want to deviate either.

Now consider party  $Q$  (indicated by the 'open' icons), whose nomination of its variable candidates  $a'_y$  ( $\circ_y$ ) also correspond to assignment  $\alpha_{\bar{p}qr}$  and who also gets all its variable and clause candidates  $b'_K$  ( $\square'_K$ ) elected. Observe that this can only be the case if  $Q$ 's clause candidates are separated from  $P$ 's variable candidates  $a_x$  with  $x \in X$  and from its own variable variables in  $D'_Y$ . Party  $Q$ 's spoiler candidate  $a''$  ( $\odot$ ), however, is not elected. As we may assume that none of its literal candidates  $d'_{\ell, i}$  ( $\heartsuit$ ) and  $d''_{\ell, j}$  ( $\triangle$ ) will be elected in equilibrium,  $Q$ 's only hope to improve would be to make sure that its spoiler candidate  $a''$  is elected along with all its variable and clause candidates. To do so,  $Q$  would have to nominate  $a''$  in a district like  $D_{r, 0}$ , which is not occupied any of  $P$ 's or  $Q$ 's candidates (see arrow ①). To get elected in  $D_{r, 0}$ , however,  $a''$  also needs all of its literal candidates  $d''_{r, i}$  to be nominated there as well to split the vote (see arrow ②). Note that then  $b'_{p\bar{q}r}$  can no longer be elected in  $D'_r$ , and  $Q$  will have to find an appropriate re-nomination that ensures all of its variable and clause candidates are still elected. Any such re-nomination by  $Q$ , however, would have to separate its clause candidates from  $P$ 's variable candidates  $a_x$  with  $x \in X$  and from its own variable candidates  $a_y$ . This, suggests an alternative assignment to  $Y$  that falsifies  $\exists X \exists Y \varphi$ , a contradiction.

Now assume that  $\exists X \exists Y \varphi$  does not hold. Then, either  $\varphi$  is satisfiable or it is not. If the latter, party  $P$  will not be able to separate its variable candidates from  $B$ 's clause candidates, and a plurality cycle with some of  $C$ 's literal candidates will be unavoidable, in a

similar way as it would in  $G_\varphi$ . If the former, we may assume that  $P$  and  $B$  separate their variable and clause candidates by nominating (all) their candidates in accordance with some assignment  $\alpha$  that satisfies  $\varphi$ . In this case, we know there is another assignment  $\alpha'$  that coincides with  $\alpha$  on  $X$  and also satisfies  $\varphi$ . By nominating its candidates in accordance with  $\alpha'$ , party  $Q$  can separate its variable and clause candidates and get them all elected. Because  $\alpha$  and  $\alpha'$  differ on some  $y \in Y$ , moreover,  $Q$  can now also get its spoiler candidate  $a''$  elected by nominating it in  $D_y$  if  $a_y$  is nominated in  $D'_y$ , and *vice versa*. Now,  $a''$  is bound to end up in a literal region where also some of  $B$ 's clause candidates are nominated. As  $a''$  behaves like  $a_y$  in the latter's absence, a plurality cycle will be unavoidable. Altogether we may conclude that no stable nomination profile exists in this case.

On this basis, we can close this section by stating the second main result, where the membership part follows from Lemma 1.

**THEOREM 2.** *DISTRICT CANDIDACY is  $\Sigma_2^P$ -complete.*

## 6 DISCUSSION AND FUTURE WORK

In this paper we initiated the study of equilibrium computation in district-based elections. Our framework and results can be extended in various ways, starting from relaxing some constraints.

Firstly, we assumed parties to only be interested in the number of their own members get elected. We may however consider parties have preferences over their own candidates, as well, or even other parties' candidates. The latter allows for a treatment of strategic candidacy into our framework, in particular the multiwinner variants [20].

We also abstracted away from voters strategically modifying their own ballot to get more preferred candidates elected. The introduction of strategic behaviour from the voters' and parties' side suggests new preference restrictions, such as single peakedness, and the exploration of district-based elections that incentivise truth telling from both sides. Also, the combination of district candidacy and strategic voting, already explored for strategic candidacy [3], is an interesting direction.

Also, campaigning in districts typically impose a cost on candidates and the results explored in the context of strategic candidacy may be of even higher relevance in our setup [19].

Our complexity analysis showed that the problem is NP-complete when the number of party candidates is bounded but becomes  $\Sigma_2^P$  in the general case. A parameterised complexity analysis is a natural follow up.  $\Sigma_2^P$  problems have been studied in the context of social choice [21, 22] and evolutionary game theory [4], whose precise connection with our work still remains to be explored.

Another important question concerns misrepresentation [23]. As parties are interested in getting their nominees elected, there may be an incentive for them to reduce misrepresentation, in the sense that nominating a stronger candidate in a district often means offering (more) voters a candidate that is ranked higher in their preferences. Understanding how equilibria affect misrepresentation is an important open challenge. Finally, the exploration of multiwinner voting rules constitutes a natural extension.

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<sup>2</sup>For presentational purposes, we dispense with the additional clauses in  $Y$ .

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