On a Voter Model with Context-Dependent Opinion Adoption

Extended Abstract

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ABSTRACT

Opinion diffusion is a crucial phenomenon in social networks, often underlying the way in which a collective of agents develops a consensus on relevant decisions. The voter model is a well-known theoretical model to study opinion spreading in social networks and structured populations. Its simplest version assumes that an updating agent will adopt the opinion of a neighboring agent chosen at random. The model allows to study, for example, the probability that a certain opinion will fixate into a consensus opinion, as well as the expected time it takes for a consensus opinion to emerge.

Standard voter models are oblivious to the opinions held by the agents involved in the opinion adoption process. We propose and study a context-dependent opinion spreading process on an arbitrary social graph, in which the probability that an agent abandons opinion a in favor of opinion b depends on both a and b. We discuss the relation of the model with existing voter models and then proceed to derive theoretical results for both the fixation probability and the expected consensus time for two opinions on an n-clique network topology, for both the synchronous and the asynchronous update modes.

KEYWORDS

opinion diffusion; voter model; fixation probability; expected consensus time

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1 INTRODUCTION

The *voter model* is a well-studied stochastic process defined on a graph to model the spread of opinions (or genetic mutations, beliefs, practices, etc.) in a population [6, 8]. In a voter model, each node maintains a state, and when a node requires updating, it will import its state from a randomly chosen neighbor. Updates can be asynchronous, with one node activating per step [8], or synchronous, with all nodes activating in parallel [6]. While the voter model on a graph has been introduced in the 1970s to model opinion dynamics, the case of a complete graph is also very well-known in population genetics where, in fact, it was introduced even earlier, to study the spread of mutations in a population [5, 9].

Mathematically, among the main quantities of interest in the study of voter models, there are the *fixation probability* of an opinion—the probability of reaching a configuration in which each node adopts the opinion—and the expected *consensus* (or *absorption*) *time*—the expected number of steps before all nodes agree on an opinion. Such quantities could in principle be computed for any n-node graph by defining a Markov chain on a set of C^n configurations, where C is the number of opinions, but such an approach is computationally infeasible even for moderate values of n. Therefore, a theoretical analysis of a voter process will often focus on obtaining upper and lower bounds for these quantities. Typically, such an analysis will still draw heavily on the theory of Markov chains [1, 7], although the synchronous and asynchronous variants often require somewhat different approaches and tools.

A limitation of the voter process is that the dynamics is oblivious to the states of both the agent u that is updating and of the neighbor that u copies its state from, and that the copying always occurs. One could easily imagine a situation (for example, in politics) where an agent holding opinion a is more willing to adopt the opinion b of a neighbor rather than to adopt opinion c; in general, the probability of abandoning opinion a in favor of opinion b might depend on both a and b. This motivates the study of *biased* voter models [2–4, 10] and in particular motivates us to introduce a voter model with an opinion adoption probability that depends on the context, i.e., on the opinions of *both* agents involved in an opinion spreading step.

2 MODEL FORMULATION

Notation. For a natural number k, let $[k] = \{0, 1, 2, \ldots, k-1\}$. If $G = (V, E)$ is a graph, we write $N(u)$ for the set of neighbors of node u in G. We write d_u for the degree of node u.

Model. We define an opinion dynamics model on networks. The parameters of the model are: i) an underlying *topology*, given by a graph G on n nodes, with symmetric adjacency matrix $A = (a_{uv})_{u,v\in[n]}$; ii) a number of *opinions* (or *colors*) $C \geq 2$; iii) an *opinion acceptance matrix* $(\alpha_{c,c'})_{c,c'\in[C]}$. The initial opinion of each agent (node) u is encoded by some $x_u^{(0)} \in [C]$.

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Table 1: Results in the unbiased case ($\alpha_{01} = \alpha_{10} = \alpha$).

Schedule	Topology	Fixation probability (of 1)	Reference	Expected consensus time	Reference
Asynchronous	Arbitrary	$\sum_u d_u x_u^{(0)} / \sum_u d_u$	[10]	T^{avoter}/α	*This paper*
Asynchronous	Clique	k/n	[5, 10]	$n^2 h(k/n)/\alpha + O(n/\alpha)$	*This paper*
Synchronous	Arbitrary	$\sum_u d_u x_u^{(0)} / \sum_u d_u$	*This paper*	$O(\beta_n T^{\text{hit}})$	*This paper*

Notes: Here $k = \sum_u x_u^{(0)}$; β_n is either $O(1)$ or $O(\ln n)$, depending on α; and $h(p) = -p \ln p - (1-p)\ln(1-p)$.

Table 2: Results in the biased case ($\alpha_{01} < \alpha_{10}$; $r = \alpha_{01}/\alpha_{10}$).

Schedule	Topology	Fixation probability (of 1)	Reference	Expected consensus time	Reference
Asynchronous	Clique	$(1 - r^{-k})/(1 - r^{-n})$	*This paper*	$\Theta(n \log n)$ $^{(*)}$	*This paper*
Synchronous	Clique	$1 - O(n^{-c})$ $^{(**)}$	[3]	$O(\log n)$ $^{(**)}$	[3]
Synchronous	Clique	$\leq k/n$	*This paper*	$\leq n/(\alpha_{10} - \alpha_{01})$	*This paper*

Notes: $^{(*)}$ universal upper bound and existential lower bound; $^{(**)}$ assumes $k = \Omega(\log n)$ nodes with initial opinion 1 and $r \geq 1 + \epsilon$, for constant $\epsilon > 0$.

For any node $u \in [n]$, we define an update process $\texttt{Update}(u)$ consisting of the following steps:

(1) **Sample**: Sample a neighbor v of u uniformly at random, i.e., according to the distribution $(a_{u1}/d_u, \dots, a_{un}/d_u)$ where $a_{uv} = 1$ if u and v are adjacent, $a_{uv} = 0$ otherwise. Here $d_u = |N(u)| = \sum_{v \in [n]} a_{uv}$ is the degree of node u.
(2) **Compare**: Compare u's opinion $c = x_u$ with v's opinion $c' = x_v$.
(3) **Accept/reject**: With probability $\alpha_{c,c'}$, set $x_u \leftarrow x_v$; in this case we say u *accepts* v's opinion. Otherwise, we say u *rejects* v's opinion.

We consider two variants of the model, differing in how the updates are scheduled. In one iteration of the *synchronous* variant, each node $u \in [n]$ applies $\texttt{Update}(u)$ in parallel. In one iteration of the *asynchronous* variant, $u \in [n]$ is sampled at random and $\texttt{Update}(u)$ is applied. We denote by $x_u^{(t)}$ the random variable encoding the opinion of node u after t iterations of either the synchronous or the asynchronous dynamics (depending on the context).

The acceptance probabilities $\alpha_{c,c'}$ are parameters of the model. We note that parameters $\alpha_{c,c'}$ with $c = c'$ are irrelevant for the dynamics, since a node sampling a neighbor of identical opinion will not change opinion, irrespective of whether it accepts the neighbor's opinion or not. Hence, to specify the opinion acceptance matrix $C(C-1)$ parameters are sufficient; we can assume that the diagonal entries equal, say, 1. In particular, when $C = 2$ it is enough to specify α_{01} and α_{10}. When $\alpha_{01} = \alpha_{10} = 1$, the model boils down to the standard voter model [6, 8].

In the rest of this work we assume $C = 2$. In this case, we say that the model is *unbiased* if the opinion acceptance matrix is symmetric, i.e., $\alpha_{01} = \alpha_{10}$, and *biased* otherwise.

Quantities of interest. The *fixation probability* of opinion 1 is the probability that there exists an iteration t such that $x_u^{(t)} = 1$ for all $u \in [n]$. The *consensus time* is the index of the first iteration t such that $x_u^{(t)} = x_v^{(t)}$ for all $u, v \in [n]$.

3 OUR CONTRIBUTION

We define and study extensions of the voter model that allow the opinion adoption probability to depend on the pair of opinions involved in an opinion spreading step. We consider both an asynchronous variant and a synchronous variant of a context-dependent voter model, with two opinions, 0 and 1. We study both the fixation probabilities and the expected consensus time. Our results are reported in Tables 1 and 2 and can be summarized as follows:

- *Unbiased / asynchronous*: we relate the expected consensus time of the process to that of a standard asynchronous voter model on arbitrary topologies; in the case of the n-clique, we get explicit, tight bounds.
- *Unbiased / synchronous*: we derive an exact value of the fixation probability, and provide an upper bound on the expected consensus time that depends on the maximum expected hitting time of the graph.
- *Biased / asynchronous*: on the n-clique, we prove that the fixation probability is $(1 - r^{-k})/(1 - r^{-n})$ where k is the number of agents initially holding opinion 1 and $r = \alpha_{01}/\alpha_{10}$. We also prove that the expected consensus time on the n-clique is $O(n \log n)$ for all values of k and $\Omega(n \log n)$ when $k = (n+1)/2$ (for odd n).
- *Biased / synchronous*: on the n-clique, the fixation probability is at least k/n if $\alpha_{01} > \alpha_{10}$, where k is the number of agents initially holding opinion 1. We prove that the expected consensus time is at most $n/(\alpha_{01} - \alpha_{10})$ in that case.

4 OPEN PROBLEMS

Due to the mathematical complexity of the model, in this paper we focus on the n-clique topology and on the case of two opinions. A generalization of the results to general network topologies seems obviously interesting but challenging. The case of more than two opinions is also wide open and would be particularly interesting already with three opinions, since one could imagine some rock-paper-scissors like dynamics if, say, the adoption probabilities satisfy $\alpha_{01} > \alpha_{10}, \alpha_{12} > \alpha_{21}, \alpha_{20} > \alpha_{02}$, so that opinion 0 is stronger than 1 but weaker than 2 and so on, which could in principle lead to a significantly larger expected consensus time.

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