Computational Complexity of Verifying the Group No-show Paradox

Extended Abstract

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ABSTRACT

The *(group) no-show paradox* refers to the undesirable situation where a group of agents has the incentive to abstain from voting to get a more favorable winner. We examine the computational complexity of verifying whether the group no-show paradox exists given agents' preferences and the voting rule. We prove that the verification problem is NP-hard to compute for commonly studied voting rules such as Copeland, maximin, single transferable vote, and Black's rule. We propose integer linear programming-based algorithms and a breadth-first search algorithm for the verification problem. Experimental results illustrate that the former work better for a small number of alternatives, and the latter work better for a small number of agents. Using these algorithms, we observe that the group no-show paradoxes rarely occur in real-world data.

KEYWORDS

Social choice; Voting axioms; Computational complexity

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1 INTRODUCTION

The *no-show paradox*, first observed by Fishburn and Brams [12], generally refers to the counter-intuitive event where a group of agents has the incentive to abstain from voting to make the winner more favorable to them. This is undesirable because when it occurs, agents can manipulate the result just by not showing up, which is much easier (thus more threatening) than strategic manipulation [1, 6, 8, 13, 14, 25] and control [2, 10, 24]. The no-show paradox also discourages voters from participating in the election, reducing turnout and undermining democracy. Unfortunately, even the *single-voter no-show paradox* always exists under a wide range of voting rules, including all Condorcet rules [5, 6, 20], scoring run-off methods [26] and all Pareto Optimal majoritarian voting rules [3].

Consequently, to understand its practical relevance, there is an extensive literature on verifying the frequency of various kinds of no-show paradox under different assumptions, such as impartial culture [17, 22, 23], single-peaked preferences [16], semi-random models [27]. Pérez [21] and Duddy [9] studied strong versions of no-show paradox's likelihood in Condorcet rules. Recently Brandt et al. [7] presented an ILP-based method for finding minimal voting paradoxes, including the no-show paradox.

We can verify the existence of a single-voter no-show paradox for many commonly-studied voting rules in polynomial time by simply enumerating the possible absentee. But we have multiple open questions for the group no-show paradox: **How likely is the occurrence of the group no-show paradox under commonly studied voting rules? What is the computational complexity of verifying the paradox?**

A high complexity of verifying the existence of a paradox will disallow voters from trying manipulation. However, a low complexity can be advantageous from a mechanism designer's perspective because it would allow us to verify whether the group no-show paradox is a significant concern in practice for a voting rule. This would help select robust voting rules against voter abstention.

Our contributions. We characterize the computational complexity of verifying group no-show paradox (GNSP) under several common voting rules: Copeland, Maximin, STV, and all Condorcetified positional scoring rules, including Black's rule. We prove that, unfortunately, the verification problem is NP-complete under all of them. To computationally solve the problem, we propose integer linear program (ILP)-based algorithms and a breadth-first search search algorithm for verifying GNSP for these voting rules. Our experiments on both synthetic data and real election data from PrefLib [19] illustrate that the ILP algorithms work better for a small number of alternatives, and the search algorithm works better for a small number of agents. We also see that no-show paradoxes rarely occur in real-world elections.

2 THE GROUP NO-SHOW PARADOX IN VOTING

For any $m \in \mathbb{N}$, let $\mathcal{A}$ denote the set of $m \geq 3$ *alternatives.* Let $\mathcal{L}(\mathcal{A})$ denote the set of all linear orders or rankings over $\mathcal{A}$. The vector of n agents' votes, denoted by P, is called a *(preference) profile.* In this paper, we focus on *resolute voting rules*, $r : \mathcal{L}(\mathcal{A})^* \to \mathcal{A}$, that map a profile to a single alternative in $\mathcal{A}$. For any profile P and any pair of alternatives a, b, let $P[a \succ b]$ denote the total number of votes in P where a is preferred to b. Let $\text{WMG}(P)$ denote the *weighted*

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majority graph of P, whose vertices are $\mathcal{A}$ and whose weight on edge $a \to b$ is $w_P(a,b) = P[a \succ b] - P[b \succ a]$. The *Condorcet winner* of a profile P, denoted by $\text{CW}(P)$, is the alternative that only has outgoing edges in $\text{WMG}(P)$.

Positional scoring rules, e.g., Plurality, Borda, do not suffer from the no-show paradox. So, we focus on other voting rules that do. Popular voting rules like Copeland, Maximin, and other Condorcetified voting rules choose the Condorcet winner when one exists, and all these rules suffer from the no-show paradox. The single transferable vote (STV) is an example of run-off or multi-round voting rules that suffer from the no-show paradox. Tie-breaking methods are applied whenever a voting rule leads to ties to get a single winner. See [4] for an exposition to voting rules.

We adopt the following definition that inherits the spirits of various definitions of the group no-show paradox problem previously given in social choice literature [11, 12, 17, 20].

Definition 1 (**Group no-show paradox (**GNSP**)**). *A group no-show paradox occurs in a profile P under a resolute voting rule r, if there exists a subset of agents $P' \subseteq P$, each of which prefers $r(P - P')$ to $r(P)$, thus giving them the incentive to abstain from voting.*

Example 1. Let $P = 6@[2 \succ 1 \succ 3] + 4@[1 \succ 3 \succ 2] + 4@[3 \succ 2 \succ 1]$. If group P' consisting of 2 votes of $[3 \succ 2 \succ 1]$ abstain from voting, then the Copeland winner (with lexicographic tie-breaking) changes from 1 to 2. Notice that $2 \succ 1$ for both agents in P'. This means that no-show paradox occurs in Copeland at P. □

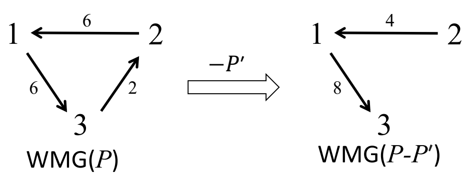

Figure 1: GNSP **under Copeland.**

3 COMPLEXITY OF VERIFYING GNSP

We discuss the computational complexity of computing the existence of group no-show paradox for Copeland, Maximin, Condocetified positional scoring rules, and STV. No-show paradoxes trivially do not occur for positional scoring rules, so we do not discuss them here. Given a voting rule r, we denote GNSP-r the computational problem that takes a profile P (n votes on m alternatives) as an input and outputs whether GNSP will occur for P.

For any fixed m and anonymous voting rule r, GNSP-r can be solved in polynomial time if the winner of r can be computed in polynomial time. This is because all possible profiles after abstentions can be enumerated in polynomial time. Unfortunately, for a variable m, we get negative results, summed up in Theorem 1.

Theorem 1. GNSP*-r is NP-complete to compute when r is Copeland, Maximin or STV where the tie-breaking mechanism is lexicographic, fixed-agent, or most popular singleton ranking-based tie-breaking.* GNSP*-r is NP-complete to compute for all Condorcetified positional scoring rules for any tie-breaking mechanism.*

The problem is in NP for all the voting rules— because given a subset of agents P', we can run the voting mechanism with and without the group to check if they have an incentive to abstain from voting. The NP-hardness is proved by reductions from RXC3, which is a restriction of EXACT 3 COVER that requires every element to be in exactly three sets and is proved to be NP-complete [15].

4 ALGORITHMS AND EXPERIMENTALS

Algorithms. We propose two types of algorithms for verifying the group no-show paradox under different circumstances.

We developed a BFS algorithm that enumerates all possible group abstentions in a breadth-first manner. In the worst case (when GNSP does not occur), the algorithm checks all possible combinations of group abstentions. and has a run-time of $[(\frac{n}{m!})^{m!} \cdot \text{Run-time}(r)]$. Although this is polynomial-time for fixed m, becomes too expensive for large n since the degree of the polynomial is very high.

For large n, we developed intelger linear program (ILP) formulations of the group no-show paradox for four different voting rules – Copeland, maximin, Black's rule, and STV. For our formulation, the variables are the number of voters voting for each linear order in $\mathcal{L}(\mathcal{A})$. The objective function minimizes the number of abstentions and the constraints are defined by each voting rule's properties. However, this formulation can be computationally expensive for high m. For example, for Copeland, for any preference profile, we need to solve $O(3^{\frac{m(m-1)}{2}})$ ILPs, each with $O(m^2)$ constraints.

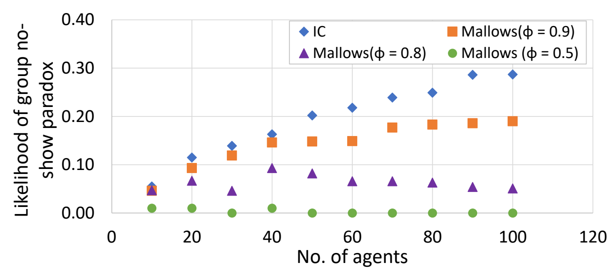

Figure 2: Likelihood of GNSP **under Copeland.**

Experiments and Results. From experiments on synthetic data, we found that, as expected, the run-time grows exponentially with the number of alternatives, m, for the ILP-based algorithm, whereas it grows exponentially with the number of agents, n, for the BFS algorithm. We used our algorithms on real election data on PrefLib. Out of the 306 observed preference profiles, only one profile each violates group participation for Copeland, Black's rule and STV, and we found no violations for Maximin. In all three occurrences of the paradox, there were high number of candidates and all agents had unique preference rankings. We repeated the experiments on synthetic data, as seen in Figure 2. We generate data using impartial culture assumptions (no correlation among the voting agents) and also by sampling from Mallows models [18] (which indicate different levels of consensus among the voting agents). It can be seen that more consensus among the agents (low ϕ value) leads to a lower likelihood of GNSP. This behavior is also seen for other voting rules as well and might explain why we see rare occurrences of GNSP for real data.

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