Equitability and Welfare Maximization for Allocating Indivisible Items

JAAMAS Track

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ABSTRACT

We study fair allocations of indivisible goods and chores in conjunction with system efficiency, measured by two social welfare functions, namely utilitarian and egalitarian welfare. To model preference, each agent is associated with a cardinal and additive valuation function. The fairness criteria we are concerned with are equitability up to any item (EQX) and equitability up to one item (EQ1). For the trade-off between fairness and efficiency, we investigate efficiency loss under these fairness constraints and establish the price of fairness. From the computational perspective, we provide an almost complete picture of the computational complexity of (i) deciding the existence of an EQX/EQ1 *and* welfare-maximizing allocation; (ii) computing a welfare maximizer among all EQX/EQ1 allocations.

KEYWORDS

Fair division; Equitability; Indivisible items; Price of Fairness

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1 INTRODUCTION

Fairness is an essential social concept and it matters in most multi-agent resource allocation problems. Agents may not accept allocations that lack fairness considerations, as each individual agent wants to be treated fairly. On the other hand, efficiency is a measurement of the utilization of resources, with which the central authority is concerned. It has been practically [2, 4, 10] and theoretically [1, 3, 11] observed that fairness and efficiency are two competing notions, in the sense that optimization on one notion may lead to bad performance on the other. It is important to explore the relationship between these two social concepts.

The underlying fairness criteria we are concerned with is *equitability* based. In an *equitable* (EQ) allocation, all agents should receive the same value. Equitability acts as an interpersonal fairness criterion and has been, in some experiments, verified to be the dominant cognitive fairness when compared to those imposed from the intrapersonal fairness criteria, such as envy-based fairness notions [5, 9]. In the allocation of indivisible items, the existence of an EQ allocation is not guaranteed, which motivates us to study two of its realistic relaxations: *equitability up to one item* (EQ1) and *equitability up to any item* (EQX). The idea of relaxing equitability through eliminating some specific items is originated by Gourvès et al. [8] in which it is named as "Near Jealousy-Freeness". Freeman et al. [6] formally define the notions of EQX and EQ1 in goods (items with non-negative value) allocation and show that both notions are satisfiable when agents have additive valuations. Then, in the setting of chores (items with non-positive value), the existence of EQX and EQ1 is proved by Freeman et al. [7].

In this work, we consider the problem of allocating indivisible items to several agents and investigate both goods and chores allocations. The main research objectives of this paper are to quantify the efficiency loss under equitability-based fairness and to study the computational complexity of deciding whether there exists an EQ1/EQX allocation that also achieves maximum social welfare. Furthermore, we study the computational complexity of computing a welfare maximizer among all EQ1/EQX allocations.

2 PRELIMINARIES

A fair division instance $\mathcal{I} = \langle [n], E, \mathcal{V} \rangle$ is composed of a set $[n]$ of agents and a set $E = \{e_1, \dots, e_m\}$ of m indivisible items, where $[n] = \{1, \dots, n\}$ for $n \in \mathbb{N}^+$. Each agent i is associated with a valuation function $v_i \in \mathcal{V}$ and $v_i : 2^E \to \mathbb{R}$. Given an item $e \in E$, we say that e is a good if for every $i \in [n]$, $v_i(e) \geq 0$ and e is a chore if for every $i \in [n]$, $v_i(e) \leq 0$. We consider the situation where all items are either goods or chores and we call $\mathcal{I}$ a *fair-goods* (resp., *fair-chores*) instance if every item is a good (resp., a chore). Throughout the paper, for every $i \in [n]$, we assume $v_i(\emptyset) = 0$ and function $v_i(\cdot)$ is *additive*, that is, $v_i(S) = \sum_{e \in S} v_i(e)$ for any $S \subseteq E$. For simplicity, instead of $v_i(\{e_j\})$, we use $v_i(e_j)$ to represent the value of item e_j on agent i. An allocation $\mathbf{A} := (A_1, \dots, A_n)$ is an n-partition of E among agents, i.e., $A_i \cap A_j = \emptyset$ for any $i \neq j$ and $\bigcup_{i \in [n]} A_i = E$. Each subset $S \subseteq E$ also refers to a *bundle* of items.

Definition 2.1. *For allocating goods, an allocation* $\mathbf{A}$ *is equitable up to one item (EQ1) if there exists* $e \in A_j$ *such that* $v_i(A_i) \geq v_j(A_j \setminus \{e\})$ *for any* $i, j \in [n]$. *For allocating chores, an allocation* $\mathbf{A}$ *is equitable up to one item (EQ1) if there exists* $e \in A_i$ *such that* $v_i(A_i \setminus \{e\}) \geq v_j(A_j)$ *for any* $i, j \in [n]$.

Definition 2.2. *For allocating goods, an allocation* $\mathbf{A}$ *is equitable up to any item (EQX) if for any* $i, j \in [n]$ *and any* $e \in A_j$ *with* $v_j(e) \neq 0$, $v_i(A_i) \geq v_j(A_j \setminus \{e\})$ *holds. For allocating chores, an allocation* $\mathbf{A}$ *is equitable up to any item (EQX) if for any* $i, j \in [n]$ *and any* $e \in A_i$ *with* $v_i(e) \neq 0$, $v_i(A_i \setminus \{e\}) \geq v_j(A_j)$ *holds.*

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Table 2: Computational complexity for fixed n

	UW Goods/Chores		EW Goods	EW Chores	
	$n = 2$	$n \geq 3$	$n \geq 2$	$n = 2$	$n \geq 3$
E(W×EQ1)	P (T4.11)	NP-complete (T4.9) pseudo-poly (T5.5)	P (T3.1)	?	NP-hard (T4.14) pseudo-poly (T5.3 & 5.6)
C(W/EQ1)	NP-hard (T4.12) pseudo-poly (T5.4)		NP-hard (T4.15) pseudo-poly (T5.1 & 5.4)		
E(W×EQX)	NP-complete (T4.7) pseudo-poly (T5.5)		P (T3.1)	NP-hard (T4.13) pseudo-poly (T5.3 & 5.6)	
C(W/EQX)	NP-hard (T4.8) pseudo-poly (T5.4)		NP-hard (T4.15) pseudo-poly (T5.2 & 5.4)		

Note: We denote by "E(W × F)" the problem of deciding whether there exists an F allocation that also maximizes W among all allocations, and denote by "C(W/F)" the problem of computing an F allocation that maximizes W among all F allocations. Abbreviations "UW" and "EW" refer to utilitarian welfare and egalitarian welfare, respectively. Abbreviation "T$x.y$" points to Theorem $x.y$ in the full version paper [12]. The complexity of E(EW×EQ1) for allocating chores to two agents is open.

Given an allocation $\mathbf{A}$, the utilitarian welfare (UW) of $\mathbf{A}$ is defined as $\text{UW}(\mathbf{A}) = \sum_{i \in [n]} v_i(A_i)$, while the egalitarian welfare (EW) of $\mathbf{A}$ is defined as $\text{EW}(\mathbf{A}) = \min_{i \in [n]} v_i(A_i)$.

We quantify the trade-off between fairness and efficiency by establishing the corresponding *price of fairness* (PoF). Informally, for the allocation of goods, PoF is the supremum ratio over all problem instances between the maximum welfare of all allocations and maximum welfare of all fair allocations. In the case of chores, PoF is the supremum ratio over all problem instances between the maximum welfare of all fair allocations and maximum welfare of all allocations.

3 SUMMARY OF THE RESULTS

On the price of fairness, we summarize the results in Table 1. In particular, in the setting of chores, the price of EQX and of EQ1 with respect to utilitarian and egalitarian welfare are both infinite. Whereas for goods, the price of EQX and of EQ1 with respect to egalitarian welfare are both 1. For utilitarian welfare, if there are two agents, the price of EQX is $3/2$ and the price of EQ1 is at least $6/5$ and at most $(\sqrt{2}+1)/2$. For general n agents, the price of EQX and of EQ1 are both at least $n-1$ and at most $3n$, asymptotically tight $\Theta(n)$.

As for the computational complexity problem, we first argue that even being restricted to the algorithmic problems we are concerned with, the chores problem may not be equivalent to the corresponding goods version, neither do the other direction.

Proposition 3.1. *For any fairness criterion $F \in \{\text{EQX}, \text{EQ1}\}$, there is no mapping $f : [-1, 0] \to \mathbb{R}_+ \cup \{0\}$ such that a fair-chores instance $\mathcal{I}^c = \langle [n], E, \mathcal{V} \rangle$ admits an F and utilitarian welfare-maximizing allocation if and only if the fair-goods instance $\mathcal{I}^g = \langle [n], E, f(\mathcal{V}) \rangle$ admits an F and utilitarian welfare-maximizing allocation.*

When concerning egalitarian welfare in goods allocation, results on the price of fairness show that there exist EQX and EQ1 allocations that achieve the optimal egalitarian welfare. We then prove that, on the contrary, when assigning chores, deciding the existence of an EQX (resp., EQ1) allocation that also maximizes the egalitarian welfare is strongly NP-hard for general n and NP-hard for fixed $n \geq 2$ (resp., $n \geq 3$).

Table 1: Prices of equitability-based fairness

	EQX	EQ1	
UW	$n = 2$: $\frac{3}{2}$ (T3.4)	$n = 2$: $\left[\frac{6}{5}, \frac{\sqrt{2}+1}{2}\right]$ (T3.5)	Goods
	$n \geq 3$: $\Theta(n)$ (T3.6)		Goods
	∞ (T3.3)		Chores
EW	1 (T3.1)		Goods
	∞ (T3.2)		Chores

Note: Interval $[a, b]$ means that the lower bound is equal to a and the upper bound is equal to b. T$x.y$ points to Theorem $x.y$ in the full version paper [12].

For optimization problems, we show that computing an EQX (or EQ1) allocation with the maximum egalitarian welfare is strongly NP-hard for general n and NP-hard for fixed $n \geq 2$ in both cases of goods and chores. Moreover, in the case of fixed n, we design pseudo-polynomial time algorithms that output an EQX or EQ1 allocation with the maximum egalitarian welfare. On the other hand, when focusing on utilitarian welfare, the computational complexity in allocating goods and chores is identical. In particular, for general n, every decision or optimization problem is strongly NP-complete and strongly NP-hard, respectively. When the number of agents n is fixed, our results are summarized in Table 2.

4 CONCLUSION

The main contribution of this work is to provide a clear picture of the efficiency loss when enforcing allocation fairness and of the computational complexity of the corresponding decision and computation problems. To move forward, given the unboundedness of the PoF in our consideration of fair and efficient allocation of chores, it is desirable to improve our current lens of the PoF to see a refined picture of the efficiency loss of a fair allocation of chores.

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