Cutsets and EF1 Fair Division of Graphs*

Extended Abstract

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ABSTRACT

A connected graph $G = (V, E)$ provides a natural context for importing the connectivity requirement of fair division from the continuous world into the discrete one. Each of n agents is allocated a share of G's vertex set V. These n shares partition V, with each required to induce a *connected* subgraph. Agents use their own valuation functions to determine the non-negative numerical values of the shares, which then determine whether the allocation is fair in some specified sense. Applications include the problem of dividing cities connected by a road network when each party wishes to drive among its allocated cities without leaving its territory.

We introduce *graph cutset*s – forbidden substructures which block allocations that are fair in the EF1 (envy-free up to one item) sense. Two parameters – *gap* and *valence* – determine blocked values of n. If G contains a cutset of gap $k \geq 2$ and valence in the interval $[n-k+1, n-1]$, then allocations that are CEF1 (connected EF1) fail to exist for n agents with certain *CM* (common monotone) valuations; an *elementary cutset* yields such a failure even for *CA* (common additive) valuations. Additionally, we provide an example (Graph G_{III} in Figure 1) which excludes both cutsets of gap at least two and CEF1 divisions for three agents even with CA valuations. We show that it is NP-complete to determine whether cutsets exist. Finally, for some graphs G we can, in combination with some new positive results, pin down G's *spectrum* – the list of exactly which values of n do/ do not guarantee CEF1 allocations. Examples suggest a conjectured common spectral pattern for all graphs.

KEYWORDS

Resource allocation; Envy-free up to one item (EF1); NP-hardness

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1 PRELIMINARIES: EF1 GRAPH DIVISIONS

Let $N = \{1, 2, \ldots, n\}$ be a finite set of *agents* and $G = (V, E)$ a connected undirected graph. We call G *traceable* if it admits a Hamiltonian path. A vertex subset $V' \subseteq V$ is *connected* if it induces a connected subgraph of G; $\mathcal{C}(V)$ is the set of all connected vertex subsets. For X any family of vertex subsets, let $\bigcup X$ denote the union of all sets in X; that is, $x \in \bigcup X$ iff $x \in S$ for some $S \in X$.

Each agent $i \in N$ has a *valuation* function $v_i : \mathcal{C}(V) \to \mathbb{R}_0^+$, with $v_i(\emptyset) = 0$; the v_i are *monotone* if for all $X, Y \in \mathcal{C}(V)$, $X \subseteq Y$ implies $v_i(X) \leq v_i(Y)$; *common* if $v_i = v_j$ for all $i, j \in N$; *arbitrary* if not required to be common; and *additive* if $v_i(V') = \sum_{x \in V'} v_i(\{x\})$ for each $i \in N$ and each $V' \in \mathcal{C}(V)$. Additive valuations are properly contained among monotone valuations, and CA (common additive) is properly contained in CM (common monotone). A *connected* allocation $A = (A_i)_{i \in N}$ of G assigns each $i \in N$ an $A_i \in \mathcal{C}(V)$, with the A_i *partitioning* V.

Envy-freeness requires $v_i(A_i) \geq v_i(A_j)$ for every pair $i, j \in N$ of agents. With indivisible objects, envy-free allocations may not exist. We use the notion of *envy-free up to one good*, aka EF1, which requires that for all $i, j \in N$, either $v_i(A_i) \geq v_i(A_j)$, or some $x \in A_j$ makes $v_i(A_i) \geq v_i(A_j \setminus \{x\})$. We use CEF1 to refer to an allocation that is connected and EF1. *Envy-freeness up to one outer good*, aka $\text{CEF1}_{\text{outer}}$, additionally requires $A_j \setminus \{x\}$ to be connected in G.

Related Work. Since its introduction [1, 4], fair division of graphs has been among the most relevant research topics of fair division with constraints. Recent work in this setting investigates different fairness concepts with respect to matters of existence and (parameterized) complexity [2, 3, 6, 7, 9, 10, 12–15].

For a path graph, Bilò et al. [3], in combination with Igarashi [11], establish that for any traceable graph G and any number n of agents with monotone valuations, there is always a $\text{CEF1}_{\text{outer}}$ allocation. It is still unknown whether any non-traceable graphs offer such a guarantee for arbitrarily many agents. However, for exactly two agents Bilò et al. [3] show that for all finite connected graphs G the following three statements are equivalent: (i) G guarantees $\text{CEF1}_{\text{outer}}$ allocations for 2 agents with monotone valuations. (ii) G guarantees $\text{CEF1}_{\text{outer}}$ allocations for 2 agents with CA valuations. (iii) G contains no tridents. (iv) G has a bipolar ordering. Here *bipolar orderings* are certain relaxations of Hamiltonian paths, while *tridents* (see the long version [5] for the precise definition) are particularly simple special cases of our graph cutsets (Definition 1). For three agents, no general characterization is known, but Igarashi and Zwicker [13] have shown that in the case of Graph G_I (Figure 1) – and of any graph obtained from it by inserting additional degree 2 vertices along its edges – $\text{CEF1}_{\text{outer}}$ allocations for three agents with monotone valuations always exist.

2 EXAMPLES OF GRAPH CUTSETS

Before we dive into the definition of cutset, we discuss examples for the graphs given in Figure 1. The cutset for graph $G_I = (V_I, E_I)$ (see also [13, Figure 11]) will be the collection $C_I = \{\{a\}, \{b\}, \{c\}\}$ of 3

*Full version available on arXiv [5].

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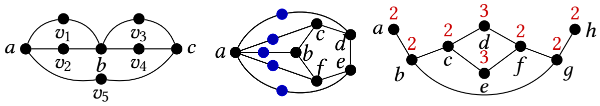

Figure 1: Left graph G_I has an elementary cutset of valence 3 and gap 2. Middle graph G_{II} has a tame cutset of valence 3 and gap 2. Right graph G_{III} has no cutset of gap 2 or greater.

singleton vertex subsets. If we delete all vertices in $\bigcup C_I$, we obtain 5 disconnected components. We use C_I to block CEF1 allocations for $n = 4$ agents as follows. Any share that overlaps more than one component, and is connected, must also contain one of the 3 vertices of $\bigcup C_I$. As $n = 4 > 3$, some *deprived* agent x gets a share disjoint from $\bigcup C_I$, hence overlapping at most one component. As $n = 4 < 5$, some *privileged* agent y gets a share that overlaps two or more components, hence overlapping $\bigcup C_I$. In this case, agent x gets at most one vertex, while y at least three, so CA valuations that assign value 1 to each of G_I's 8 vertices results in x envying y by more than one object.[1]

A cutset member is called *type I* if it is a singleton and *type II* if not. All C_I members were type I, making it an *elementary* cutset. Graph G_{II}'s cutset $C_{II} = \{\{a\}, \{b, c, d, e, f\}\}$ has *one* type-II member, making it a *tame* cutset. The induced graph on $V_{II} \setminus \bigcup C_{II}$ again has 5 disconnected components. A connected share overlapping 2 components must contain a, or contain two from $\{b, c, d, e, f\}$, so at most 3 agents have "tickets" to connect two components; with 4 agents, someone gets no ticket. By assigning value $\frac{1}{2}$ to each vertex in $\{b, c, d, e, f\}$ and value 1 to the others, a version of the deprived agent argument goes through. Theorem 1 applies a version of this argument whenever the number of agents lies strictly between the number τ of "tickets" (called the *valence*, see Definition 1) and the number $\tau + r$ of sets in a certain partition of $V_{II} \setminus \bigcup C$,[2] with the difference r between these numbers termed the *gap*. But for the non-tame case, the counterexample valuations constructed are CM, not CA. Cutsets arose as generalizations of the tridents from Bilò et al. [3].

3 MAIN DEFINITION AND THEOREM

Type-II members complicate our definition for *generalized* cutset. For elementary cutsets, the definition greatly simplifies. These notions differ from the *graph cutsets* of graph theory, which are sets of edges that disconnect the graph when deleted.

Definition 1 (generalized cutset; in short, cutset). *For $G = (V, E)$ a finite connected graph, let*

- *$C = \{C_1, C_2, \ldots, C_t\}$ be a family of t pairwise disjoint, nonempty subsets of V, each inducing a connected subgraph,*
- *$\tau = (\tau_1, \tau_2, \ldots, \tau_t)$ be a sequence of natural numbers, with τ_j called C_j's* pass-through number *and the sum $\Sigma\tau$ of all τ_j the* valence*,*
- *$G \setminus C$ be the subgraph of G induced by $V \setminus \bigcup C$, and*
- *$H = \{H_1, H_2, \ldots, H_{\Sigma\tau+r}\}$, with $r \geq 2$, be a partition of $V \setminus \bigcup C$ that is* independent*, meaning that whenever two vertices belong to different members of the partition, they are non-adjacent.*

Assume, in addition, that

- *for each C_i and H_j there is at most one vertex $s_{i,j}$ in C_i adjacent to any vertices in H_j, referred to as the* contact vertex *for C_i and H_j,*
- *each $C_j \in C$ is either a "type-I member" containing one vertex, or a "type-II member" containing more than one,*
- *the type-II members of C form an independent family, and*
- *for each type-II member C_i, there are $2\tau_j + 1$ sets H_j each admitting a contact vertex $s_{i,j} \in C_i$, and these vertices are distinct.*

Then C is a cutset of valence $\Sigma\tau$ and gap $r \geq 2$, with witness H. Such a cutset is elementary *if it contains no type-II members, and is* tame *if it contains at most one.*[3]

Theorem 1 (Main Theorem). *Let $G = (V, E)$ be a finite, connected graph. Suppose $C = \{C_1, C_2, \ldots, C_t\}$ is a cutset for G, of gap $r \geq 2$ and valence $\Sigma\tau$, with witness $H = \{H_1, H_2, \ldots, H_{\Sigma\tau+r}\}$. Then for each integer n lying within the "critical interval" $\Sigma\tau < n < r + \Sigma\tau$,*

- *there exist CM valuations for n agents, under which no* CEF1 *allocations exist (whence no* $\text{CEF1}_{\text{outer}}$ *allocations exist);*
- *if C is tame, there exist CA valuations for n agents, under which no* CEF1 *allocations exist (whence no* $\text{CEF1}_{\text{outer}}$ *allocations exist).*

A counterexample. Theorem 1, along with a result by Bilò et al. [3] (see Related Work) implies that cutsets (of gap $\geq$ 2) are forbidden substructures for traceable graphs. The converse is not true; Graph G_{III} is not traceable, yet fails to admit any cutset of gap at least two. Moreover, the numbers on the vertices provide CA valuations for which no $\text{CEF1}_{\text{outer}}$ allocation exists for $n = 3$ agents.

4 COMPLEXITY AND A CONJECTURE

It is well-known that traceable graphs are NP-hard to detect [8]. In the following, we show the same for cutsets.

Theorem 2. *It is NP-complete to decide whether a graph admits an elementary cutset (resp. cutset) of valence t and gap ≥ 2.*

Theorem 3. *For all connected graphs $G = (V, E)$ and n agents with monotone valuations, if $n \geq |V| - 1$, or if $n = |V| - 2$ and no three vertices a, b, u exist such that u is a's only neighbor and b's only neighbor, then there always exists an* $\text{CEF1}_{\text{outer}}$ *allocation.*

Theorem 3 helps to show that the *spectrum* of the 8 vertex graph G_I is ⟨yes, yes, yes, no, YES⟩. Here the first 3 *yes* entries signify that G_I guarantees existence of $\text{CEF1}_{\text{outer}}$ allocations for $n = 1, 2,$ and 3 agents, the *YES* in the fifth spot indicates the same is true for all $n \geq 5$, and the *no* in spot 4 indicates that CEF1 allocations for $n = 4$ agents fail to exist for some choice of valuations. Similar patterns for the spectra of other graphs suggest:

Conjecture 1. *The spectrum of any finite graph G consists of an infinite* yes *string, interrupted by a (possibly empty) finite* no *string.*

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[1] When components have several vertices, assign value 1 to one and 0 to the rest.

[2] However, a set in the partition might equal a union of several components.

[3] Requirements that cutset members induce connected subgraphs and pass-through numbers $2\tau_j + 1$ be odd are unused in the Theorem 1 proof, but reduce the search space for cutsets. If some number n of agents lies in the critical interval of a cutset C satisfying a version of Definition 1 with these requirements dropped, then n lies in the critical interval of a second cutset C' satisfying the full definition.

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