Candidate Nomination for Condorcet-Consistent Voting Rules

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ABSTRACT

Consider elections where the set of candidates is partitioned into parties, and each party must nominate exactly one candidate. The Possible President problem asks whether some candidate of a given party can become the winner of the election for some nominations from other parties. We perform a multivariate computational complexity analysis of Possible President for a range of Condorcet-consistent voting rules, namely for Copeland$^\alpha$ for $\alpha \in [0, 1]$ and Maximin. The parameters we study are the number of voters, the number of parties, and the maximum size of a party. For all voting rules under consideration, we obtain dichotomies based on the number of voters, classifying NP-complete and polynomial-time solvable cases. Moreover, for each NP-complete variant, we determine the parameterized complexity of every possible parameterization with the studied parameters as either (a) fixed-parameter tractable, (b) W[1]-hard but in XP, or (c) para-NP-hard, outlining the limits of tractability for these problems.

KEYWORDS

elections; parties; candidate nomination; Copeland voting; Maximin voting; parameterized complexity

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1 INTRODUCTION

Political elections are always preceded by a turbulent phase where parties select their nominated candidates for the upcoming election. Clearly, this process has a great influence on the outcome of the election, and therefore it is only natural that political parties engage in all kinds of strategic behavior when choosing their nominees. We focus on the case which models presidential elections in the sense that each party needs to nominate exactly one person among its possible candidates for presidency.

A naive approach would expect each party to simply choose its "best" candidate—however, in practice it is rarely the case that there is a single candidate that can be considered the best in all scenarios. Indeed, a given party may find that different candidates have different chances of winning the upcoming election depending on the nominees of the remaining parties. Parties may elect their nominees through primaries (an approach studied by Borodin et al. [5]), but a more careful process may take into account the estimated preferences of all voters over the possible nominees, and not only the preferences of party members.

Following the formal model of candidate nomination proposed by Faliszewski et al. [10], we assume that the preferences of all voters over all potential candidates are known, and in the *reduced election* obtained as a result of each party nominating a unique candidate, the preferences of each voter over these nominees are simply the restriction of its preferences over the whole pool of candidates. Faliszewski et al. asked two natural questions: the Possible President problem asks whether a given party can nominate some candidate c in such a way that c can become the winner of the election for *some* nominations from the remaining parties, and the Necessary President problem asks whether some nominee c of the given party will be the winner irrespective of all other nominations.

In this paper we study the Possible President problem in election systems that use some *Condorcet-consistent* voting rule. A candidate that defeats all other candidates in a pairwise comparison is called the *Condorcet winner*, and voting rules that always choose the Condorcet winner if it exists are said to be *Condorcet-consistent*. We focus our investigations on the Condorcet-consistent voting rules Maximin and Copeland$^\alpha$ for $\alpha \in [0, 1]$.

Condorcet-consistent voting rules are widely used in sports competitions, but have also been applied by e.g., the Pirate Party in Sweden and in Germany, and various organizations such as Debian, Gentoo Foundation, and Wikimedia [20]. Foley [13] has suggested to use Condorcet-consistent round-robin voting for primary elections, followed by a general election between the top two candidates, to overcome the serious flaw in US presidential elections that the winner may not be the preferred candidate of the majority of voters.

Related Work. Faliszewski et al. [10] dealt only with Plurality, arguably the simplest type of elections, and derived several NP-hardness results for both Possible and Necessary President. They also showed that when preferences are single-peaked, Necessary President can be decided in polynomial time. By contrast, they found that Possible President remains NP-complete even for single-peaked preferences, though becomes tractable if the candidates of each party appear consecutively on the societal axis.

Misra [16] extended the results of Faliszewski et al. by studying the parameterized complexity of Possible President. She examined the number t of parties as the parameter, and proved that the problem is W[2]-hard and in XP, and becomes fixed-parameter tractable (FPT) with parameter t when restricted to 1D-Euclidean preference profiles. She also strengthened previous results by proving that Possible President for Plurality is NP-hard even if all parties have size at most two, and preferences are both single-peaked and single-crossing; hence, the problem is para-NP-hard when parameterized by the size of the largest party even on a very restricted domain. Misra asked whether Possible President for

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# voters	Copeland$^\alpha$ classical	Copeland$^\alpha$ param. t	Maximin classical	Maximin param. t
$n = 2$	$\alpha = 1$: P $\alpha < 1$: NP-c (T3.1,T3.2)	– open	P (T4.1)	–
$n = 3$	NP-c (T3.3)	open	P (T4.2)	–
$n \geq 4$ even	NP-c (T3.2,T3.6)	W[1]-h, XP (T3.8)	NP-c (T4.3)	FPT (T4.4)
$n \geq 5$ odd	NP-c (T3.3)	W[1]-h, XP (T3.9)	NP-c (T4.3)	FPT (T4.4)

Table 1: Summary of our results on the classical and parameterized complexity of the Possible President problem. Our parameterized results for NP-hard cases consider parameter t, the number of parties. "NP-c" and "W[1]-h" stand for "NP-complete" and "W[1]-hard", respectively. All our NP-completeness results hold for maximum party size $\sigma = 2$.

Plurality is FPT when parameterized by the number of voters; this questions has been answered negatively by Schlotter et al. in [19].

Possible President for voting rules other than Plurality have been first treated by Cechlárová et al. in [6]. Namely, they dealt with positional scoring rules (ℓ-Approval, ℓ-Veto, and Borda) and with Condorcet-consistent rules Copeland, Llull, and Maximin. They proved that Possible President is NP-hard for each of these rules, even when the maximum size of a party is two; they left the complexity for Copeland$^\alpha$ with $\alpha \in (0, 1)$ open.

Schlotter et al. [19] obtained results concerning the parameterized complexity of Possible President for several classes of positional scoring rules, including Borda and nontrivial generalizations of ℓ-Approval and ℓ-Veto. The parameters they examined were the number of voters, the number of voter types, the number of parties, the maximum size of a party and their combinations.

Further results concerning elections with parties that nominate candidates have been provided by Lisowski [14]. He considered directed graphs called tournaments whose vertices correspond to candidates, and each directed arc (a, b) indicates that a majority of voters prefers candidate a to candidate b. Among others, Lisowski observed that it is possible to check whether a given party has a possible Condorcet winner in polynomial time, while the problems to decide whether a Nash equilibrium exists in the associated game and whether a given party has a Condorcet winner in some Nash equilibrium are NP-complete.

For a broader view on research related to candidate nomination, we refer the reader to the full version of our paper [18].

Our contribution. We perform a detailed multivariate complexity analysis using the framework of parameterized complexity for Possible President for two types of Condorcet-consistent voting rules: Copeland$^\alpha$ for every $\alpha \in [0, 1]$ and Maximin. Our parameters are the following: the number n of voters, the number t of parties, and the size σ of the largest party. Table 1 summarizes our results.

For Copeland$^\alpha$ elections with $\alpha \in [0, 1]$, we obtain a complete computational dichotomy for the complexity of Possible President as a function of the number of voters:

Theorem 1.1. *Let n be a fixed integer and $\alpha \in [0, 1]$. Then Possible President for Copeland$^\alpha$ is NP-complete when restricted to instances with n voters and maximum party size $\sigma = 2$, if*

(a) $n \geq 3$, or
(b) $n = 2$ and $\alpha < 1$.

By contrast, Possible President for Copeland^1 (i.e., Llull) restricted to instances with 2 voters is polynomial-time solvable.

It transpires that Possible President for Copeland$^\alpha$ for arbitrary $\alpha \in [0, 1]$ is para-NP-hard when parameterized by $n + \sigma$, i.e., both the number of voters and the maximum party size. We strengthen this result by showing that parameterizing the problem with t, the number of parties, the problem remains W[1]-hard even if the number of voters is a constant $n \geq 4$. Since the problem is easily solvable in $\sigma^t n^{O(1)}$ time [19], this yields a classification of all parameterized (NP-hard) variants of Possible President for Copeland$^\alpha$, with parameters chosen arbitrarily from $\{n, \sigma, t\}$, as either (i) FPT, (ii) W[1]-hard and in XP, or (iii) para-NP-hard.

We remark that despite this complete classification, we leave the computational complexity open for certain constant values of n; namely, we could not resolve the parameterized complexity of Possible President for Copeland$^\alpha$ for parameter t when $n \in \{2, 3\}$.

For the Maximin voting rule, we again obtain a complete dichotomy with respect to the number n of voters:

Theorem 1.2. *Let n be a fixed integer. Then Possible President for Maximin voting rule for instances with n voters is*

(a) polynomial-time solvable if $n \leq 3$;
(b) NP-complete if $n \geq 4$, even for maximum party size $\sigma = 2$.

Contrasting the Copeland$^\alpha$ voting rule, we show that Possible President for Maximin is FPT when parameterized by the number t of parties. This tractability result is achieved by a reduction of our problem to a special polynomial-time solvable version of the Partitioned Subdigraph Isomorphism problem. Thus, our results for Maximin yield a complete classification of all parameterized (NP-hard) variants of the problem as either FPT or para-NP-hard. In fact, we settle the complexity of the problem for each variant where n and σ both may be restricted to *arbitrary* fixed integers as either NP-complete and FPT with t, or polynomial-time solvable.

Techniques. Our algorithmic results use standard techniques from parameterized complexity and algorithmic graph theory. Our hardness results rely on intricate constructions, and we also develop the technique of using so-called *flat elections* with three voters and m candidates where each candidate defeats exactly $\frac{m-1}{2}$ candidates; this method might be of independent interest.

Results marked by ($\star$) have their proofs in the full version [18].

2 PRELIMINARIES

We use the notation $[i] = \{1, 2, \ldots, i\}$ for each positive integer i.

We assume familiarity with basic graph theory and the framework of parameterized complexity. Besides providing all necessary definitions in the full version [18], we refer the reader to the books [7, 9] for an introduction into parameterized complexity, and to the books [2, 8] for the standard notation on graphs we adopt.

Elections. An election $\mathcal{E} = (C, V, \{\succ_v\}_{v \in V})$ consists of a finite set C of candidates, a finite set V of voters, and the preferences of voters over candidates. We assume that the preferences of each voter v are represented by a strict linear order $\succ_v$ over C, where $c \succ_v c'$ means that voter v *prefers* candidate c to candidate c'. We denote the set of all elections over a set C of candidates by $\mathbb{E}_C$. A *voting rule* $f : \mathbb{E}_C \to 2^C$ chooses a set of *winners* of the election.

Our model also includes a partition $\mathcal{P} = \{P_1, \dots, P_t\}$ of the set C of candidates; each set P_j is interpreted as a *party* that has to decide whom among its potential candidates to nominate for the election.

Formally, a *reduced election* arises after all parties have nominated a unique candidate, leading to a reduced candidate set $C' \subseteq C$ such that $|C' \cap P_j| = 1$ for each $j \in [t]$. We can then define the reduced election as $\mathcal{E}_{C'} = (C', V, \{\succ'_v\}_{v \in V})$ where the preference relation $\succ'_v$ of each voter $v \in V$ is the restriction of her original preference relation $\succ_v$ to C'.

Now we formulate our problem of interest, as introduced in [10].

Problem **Possible President** for voting rule f:
Input: An election $\mathcal{E} = (V, C, \{\succ_v\}_{v \in V})$ with a set V of voters and a set C of candidates, a partition $\mathcal{P}$ of C into parties, and a distinguished party $P^\star \in \mathcal{P}$.
Question: Is there a candidate $p \in P^\star$ such that for some nominations of other parties leading to a reduced candidate set C', p is the *unique* winner of the reduced election $\mathcal{E}_{C'}$ according to f?

Notice that we consider the *unique winner model*, i.e., we aim for a set of nominations that yield $f(\mathcal{E}_{C'}) = \{p\}$ for the candidate p nominated by the distinguished party in the reduced election $\mathcal{E}_{C'}$.

Voting rules. In this paper we shall concentrate on two Condorcet-consistent rules. For two candidates $a, b \in C$, we let $N_\mathcal{E}(a, b)$ denote the number of voters who prefer candidate a to candidate b in election $\mathcal{E}$; we shall omit the subscript when $\mathcal{E}$ is clear from the context. If $N_\mathcal{E}(a, b) > N_\mathcal{E}(b, a)$ we say that candidate a *defeats* candidate b in $\mathcal{E}$; if $N_\mathcal{E}(a, b) = N_\mathcal{E}(b, a)$ and $a \neq b$, then candidates a and b are *tied* in $\mathcal{E}$. The *Condorcet winner* is the candidate that defeats all other candidates; a voting rule is *Condorcet consistent*, if it always selects the Condorcet winner whenever it exists.

The *Copeland*$^\alpha$ *voting rule*, as defined by Faliszewski et al. [11], assigns to some candidate a a score of 1 for each candidate defeated by a, and a score of α for each candidate tied with a, so the Copeland$^\alpha$-score of a is $\mathsf{Cpl}^\alpha_\mathcal{E}(a) = \sum_{b \text{ defeated by } a} 1 + \sum_{b \text{ tied with } a} \alpha$ in an election $\mathcal{E}$. The winners of $\mathcal{E}$ are all candidates with the maximum score. The voting rule obtained for $\alpha = 1$ is called the *Llull* rule, and we refer to the case $\alpha = 0$ as the *Copeland* rule.

In the *Maximin voting rule*, the Maximin-score of candidate a in election $\mathcal{E}$ over candidate set C is $\mathsf{MM}_\mathcal{E}(a) = \min_{b \in C \setminus \{a\}} N_\mathcal{E}(a, b)$, and the winners of $\mathcal{E}$ are again the candidates with maximum score.

Notice that Copeland$^\alpha$ as well as Maximin winners can be computed efficiently for any election. Therefore it is easy to see that Possible President for these voting rules belongs to the class NP.

3 COPELAND$^\alpha$ VOTING RULE

If there are only two voters, in Section 3.1 we show that Possible President for Copeland$^\alpha$ is polynomially solvable if $\alpha = 1$, (Theorem 3.1), but NP-hard if $\alpha < 1$ (Theorem 3.2).

For three voters, we show in Section 3.2 that Copeland is NP-complete (Theorem 3.3). If the number of voters is odd, then no ties occur, and hence this result holds for Copeland$^\alpha$ for any $\alpha \in [0, 1]$. The proof is quite involved, and provides a reduction from a special variant of the NP-complete problem Maximum Matching with Couples, using the crucial notion of *flat* elections.

The case with four or more voters is treated shortly in Section 3.3.

We address the complexity of Possible President for Copeland when parameterized by the number of parties in Section 3.4.

3.1 Two Voters

Let us first show that Possible President for the Llull voting rule is easy if there are only two voters. The key observation that yields tractability is that the "defeat" relation is transitive for two voters:

Observation 1. *In an election with two voters, if candidate a defeats candidate b, and b defeats candidate c, then a also defeats c.*

Proof. Since b must follow a in the preference lists of both voters, and c must follow b in both lists too, we immediately know that c follows a in the preference list of both voters. □

Theorem 3.1 (★). *Possible President for the Llull voting rule is polynomial-time solvable if there are only two voters.*

Proof sketch. Using Observation 1, one can prove that according to the Llull rule, some candidate p can be a unique winner in a reduced election $\mathcal{E}$ if and only if p defeats every other nominee: intuitively, assuming that p is the unique winner because every nominee other than p is defeated by some other nominee, we arrive at a cycle in the defeat relation, a contradiction showing that p can become the unique winner only by defeating all nominees.

This offers a quadratic-time algorithm to solve Possible President for Llull voting with two voters: we check for each candidate p in the distinguished party P whether p can become the unique winner, which happens if and only if every other party contains at least one candidate that is defeated by p. ⊿

By contrast, a reduction from 3-Coloring shows that Copeland$^\alpha$ for $\alpha < 1$ is intractable already for two voters, even if $\sigma = 2$.

Theorem 3.2 (★). *For each $\alpha \in [0, 1)$, Possible President for Copeland$^\alpha$ is NP-complete even for instances with two voters and maximum party size $\sigma = 2$.*

3.2 Three Voters

As already mentioned, for an odd number of voters no two candidates can be tied, so the value of α is irrelevant, and the Copeland and Llull voting rules coincide. We show the following.

Theorem 3.3 (★). *Possible President for Copeland is NP-complete even for three voters and maximum party size $\sigma = 2$.*

To show Theorem 3.3, we will reduce from a special case of an NP-complete problem Maximum Matching with Couples, described in Section 3.2.1. We present the most important ingredient of the reduction, the notion of *flat elections* in Section 3.2.2, and follow with a sketch of the reduction in Section 3.2.3.

3.2.1 A special case of Maximum Matching with Couples. We are going to reduce from a variant of the following problem called Maximum Matching with Couples. This problem involves a set S

of *singles*, a set C of *couples*[1] and a set R of rooms. Each room has capacity 2, meaning that it can accommodate either a couple or at most two singles. Moreover, we need to match everyone to a room that they find *acceptable*, where acceptability is described by a bipartite graph $G = ((S \cup C) \uplus R, E)$. A *complete matching*[2] in G is then an edge set $M \subseteq E$ that contains exactly one edge incident to each vertex in $S \cup C$ and satisfies $|M(r) \cap S| + 2|M(r) \cap C| \leq 2$ for each room $r \in R$, where $M(r) = \{x \in S \cup C \colon rx \in M\}$ denotes the set of singles and couples *matched to* r. It is known that the following problem is NP-complete [3, 4].

Problem **Maximum Matching with Couples**:
Input: Sets S, C, and R of singles, couples and rooms, respectively, and a bipartite graph $G = ((S \cup C) \uplus R, E)$.
Question: Is there a complete matching in G?

We shall use a special case of Maximum Matching with Couples as specified in Theorem 3.4. The proof of its NP-completeness relies on a series of simple reduction rules that transform any instance into an equivalent one, achieving the properties required in Theorem 3.4 step by step; see the full version [18].

Theorem 3.4 (★). *Maximum Matching with Couples remains* NP*-complete even if* $|R| = |S|/2 + |C|$*, and*

- *each vertex in the input graph has degree* 2 *or* 3*, and*
- *each room adjacent to both singles and couples is adjacent to exactly two singles and one couple.*

3.2.2 Flat elections with three voters. Working towards a reduction from Maximum Matching with Couples to Possible President for Copeland voting with three voters, we next present a construction for an election $\mathcal{E}_q$ over 3^q candidates for some $q \in \mathbb{N}^+$ and with three voters, in which every candidate defeats the same number of candidates. We will call such elections *flat*, i.e., an election is flat if all candidates receive the same Copeland-score. An election with m candidates where m is odd can only be flat if each candidate defeats $\frac{m-1}{2}$ other candidates. To see this, consider the tournament underlying the election: clearly, we can only have all out-degrees equal to some d, if the tournament has $m \cdot d$ arcs, i.e., $m \cdot d = \binom{m}{2}$.

We propose a recursive construction for $\mathcal{E}_q$ in Definition 3.5.

Definition 3.5. Let the candidate set of $\mathcal{E}_1$ be $C_1 = \{\underline{a}, \underline{b}, \underline{c}\}$, and let w, w', and w'' be our three voters with preferences

$$w : \underline{a}, \underline{b}, \underline{c};$$
$$w' : \underline{c}, \underline{a}, \underline{b};$$
$$w'' : \underline{b}, \underline{c}, \underline{a}.$$

Notice that $\underline{a}$ defeats $\underline{b}$, $\underline{b}$ defeats $\underline{c}$, and $\underline{c}$ defeats $\underline{a}$. Therefore, each of the candidates obtains a Copeland$^\alpha$-score of 1.

For $q \geq 1$, we are going to reuse the candidate set C_q of the election $\mathcal{E}_q$ to construct the candidate set C_{q+1} of $\mathcal{E}_{q+1}$ by introducing three copies of each candidate $c \in C_q$ which will be denoted by $c \odot 1$, $c \odot 2$, and $c \odot 3$. Let $L_q(w)$, $L_q(w')$, and $L_q(w'')$ denote the preference lists of voters w, w', and w'', respectively, in $\mathcal{E}_q$. For a list L of candidates from C_q and each $h \in [3]$, let us denote by $L \odot h$ the list obtained from L by replacing each candidate c in L by its h-th copy $c \odot h$. Using this notation, we are now ready to define the preferences of the voters in $\mathcal{E}_{q+1}$:

$$\begin{array}{llll} w: & L_q(w) \odot 1, & L_q(w) \odot 2, & L_q(w) \odot 3; \\ w': & L_q(w') \odot 3, & L_q(w') \odot 1, & L_q(w') \odot 2; \\ w'': & L_q(w'') \odot 2, & L_q(w'') \odot 3, & L_q(w'') \odot 1. \end{array} \tag{1}$$

Notice that each candidate in $\mathcal{E}_{q+1}$ is then of the form

$$(((x \odot h_1) \odot h_2) \cdots \odot h_{q-1}) \odot h_q \tag{2}$$

for some $x \in \{\underline{a}, \underline{b}, \underline{c}\}$ and indices $h_1, h_2, \ldots, h_q \in [3]$.

We will say that two candidates c and c' in $\mathcal{E}_{q+1}$, having the form (2) for x and x' in $\{\underline{a}, \underline{b}, \underline{c}\}$ and indices $h_1, \ldots, h_q$ and $h'_1, \ldots, h'_q$ from [3], respectively, *belong to the same group at level* q' for some $q' \in [q]$, if $h_i = h'_i$ for each $q' \leq i \leq q$; accordingly, we define a *q'-level group* as a maximal set of candidates that belong to the same group at level q'. Notice that restricting the election $\mathcal{E}_{q+1}$ to a q'-level group, we obtain a copy of the election $\mathcal{E}_{q'}$.

In particular, restricting $\mathcal{E}_{q+1}$ to a q-level group, that is, to the set of candidates contained in $L_q(w) \odot h$ for some $h \in [3]$, we obtain a copy of the election $\mathcal{E}_q$. Observing the preferences of the voters as given in (1), the following facts are immediate:

Observation 2. *For each* $q \in \mathbb{N}$*, the election* $\mathcal{E}_{q+1}$ *has the following properties:*

- *each candidate in* $L_q(w) \odot 1$ *defeats all candidates in* $L_q(w) \odot 2$*, and is defeated by all candidates in* $L_q(w) \odot 3$*;*
- *each candidate in* $L_q(w) \odot 2$ *defeats all candidates in* $L_q(w) \odot 3$*, and is defeated by all candidates in* $L_q(w) \odot 1$*;*
- *each candidate in* $L_q(w) \odot 3$ *defeats all candidates in* $L_q(w) \odot 1$*, and is defeated by all candidates in* $L_q(w) \odot 2$*.*

Furthermore, for each $h \in [3]$ *and each* $c, c' \in C_q$*, candidate* $c \odot h$ *defeats candidate* $c' \odot h$ *in* $\mathcal{E}_{q+1}$ *if and only if* c *defeats* c' *in* $\mathcal{E}_q$*.*

By Observation 2, $\mathcal{E}_{q+1}$ for some $q \in \mathbb{N}^+$ is flat if and only $\mathcal{E}_q$ is flat. Since $\mathcal{E}_1$ is flat, we obtain the following consequence.

Observation 3. *For each integer* $q \geq 1$*, every candidate in* C_q *defeats* $\frac{|C_q|-1}{2} = \frac{(3^q-1)}{2}$ *candidates in* $\mathcal{E}_q$*, so* $\mathcal{E}_q$ *is a flat election with* 3^q *candidates. Moreover, no candidate is preferred to another candidate by all three voters in* $\mathcal{E}_q$*.*

3.2.3 Reduction for Theorem 3.3. We present a reduction from the variant of Maximum Matching with Couples described in Theorem 3.4. Let $G = ((S \cup C) \uplus R, E)$ be the input graph.

High-level description. The main ideas of the reduction are the way flat elections are used. First, we need a large enough set T of *teams*, over which we have a flat election involving three voters. Each team in T will be either a single, a couple or its copy, a room, or a dummy, and will be eventually be replaced by a set of candidates, depending on its type. We will also add a set $A \cup B$ of *simple candidates*, and we fix a simple candidate a_1 to form the distinguished singleton party in the constructed instance.

Based on our flat election over T, we do three modifications: (i) we insert the simple candidates, (ii) we substitute each team in T with the corresponding candidate lists, and (iii) we move our distinguished candidate a_1 "to the left" so that it gains one extra point in the election. The crux of the reduction is to ensure that in

[1] Although in the context of elections C denotes the set of candidates, this slight clash of notation will not cause any confusion.
[2] Note that we do not require M to be a matching in the classic graph-theoretic sense, since we allow edges in M to share endpoints in R.

the obtained election, a_1 can become the unique winner if and only if restricting the election to the *relevant* candidates (those that are associated with some team in T) yields a flat election. By carefully designing the candidate set corresponding to each team and their ordering within the preference lists (used during the substitution step), we will ensure that the relevant candidates can form a flat election if and only if our instance of Maximum Matching with Couples admits a complete matching.

Candidates and parties. First, we define a party $P_r = \{r, r'\}$ for each room $r \in R$. Next, for each vertex $p \in S \cup C$ adjacent to r in G, we introduce a party $P_p^r = \{p^r, \neg p^r\}$. Additionally, we define two candidates p and p' for each $p \in S \cup C$; if p has degree 3 in G, then these two candidates form a single party, and if p has degree 2 in G, then p and p' both form their own singleton party. This way, we associate four parties with each single $s \in S$:

- $P_s^{r_1}, P_s^{r_2}, \{s\}, \{s'\}$ if $N_G(s) = \{r_1, r_2\}$,
- $P_s^{r_1}, P_s^{r_2}, P_s^{r_3}, \{s, s'\}$ if $N_G(s) = \{r_1, r_2, r_3\}$

where $N_G(v)$ denotes the neighborhood of a vertex v in G. Similarly, there are four parties associated with each couple $c \in C$:

- $P_c^{r_1}, P_c^{r_2}, \{c\}, \{c'\}$ if $N_G(c) = \{r_1, r_2\}$;
- $P_c^{r_1}, P_c^{r_2}, P_c^{r_3}, \{c, c'\}$ if $N_G(c) = \{r_1, r_2, r_3\}$.

Next, for each couple $c \in C$, we introduce a copy $\hat{x}$ for each candidate x associated with the couple c, yielding a candidate set $\{\hat{c}^r, \neg\hat{c}^r : r \in N_G(c)\} \cup \{\hat{c}, \hat{c}'\}$. We write $\hat{C} = \{\hat{c} : c \in C\}$. With each $\hat{c} \in \hat{C}$ we associate the parties $P_{\hat{c}}^r = \{\hat{x} : x \in P_c^r\}$ for each $r \in N_G(c)$, plus one or two parties formed by $\hat{c}$ and $\hat{c}'$, depending on whether c has degree two or three in G, so that altogether there are four parties associated with $\hat{c}$ (as for c). For practical purposes, we extend the notation by setting $N_G(\hat{c}) := N_G(c)$ for each $c \in C$.

We also fix an arbitrary set D of *dummy teams* whose size is the smallest non-negative integer for which $\rho := |R|+|S|+2|C|+|D| = 3^q$ for some $q \in \mathbb{N}^+$, and introduce candidates a_d, b_d, c_d for each $d \in D$, each of them forming its own singleton party. Since for each positive integer n there is a power of 3 in the interval $[n, 3n]$ (this is easily shown by induction on n), we get $|D| \leq 2(|S| + 2|C| + |R|)$.

We call the candidates defined so far *relevant candidates*, and denote their set as X. We further define *simple candidates* $a_1, \ldots, a_{3\rho}$ and $b_1, \ldots, b_{3\rho}$, each of them forming its own singleton party. We will write $A = \{a_1, \ldots, a_{3\rho}\}$ and $B = \{b_1, \ldots, b_{3\rho}\}$. Notice that the maximum party size is $\sigma = 2$ in G, and the number of parties is $|R| + 4|S| + 8|C| + 3|D| + 6\rho = 9\rho$. Our distinguished party is $\{a_1\}$.

Teams and their lists. We refer to the set $T = S \cup C \cup \hat{C} \cup R \cup D$ as the set of *teams*. To define the preferences of our voters, v, v', and v'', we introduce for each team $t \in T$ three lists that we call *team lists* and denote by F_t, F'_t, and F''_t. Each of these three lists contains the same set candidates that we associate with t.

Let us start with defining the team lists for each room team $r \in R$. First, if room r is adjacent to singles s_1 and s_2 and a couple c in G, then we set its team list according to (3) below (to the left). Second, if room r is adjacent to two or three singles, s_1, s_2 and possibly s_3, and no couples in G, then we set its team list as in (4).

if $N_G(r) = \{s_1, s_2, c\}$:

$$\begin{aligned} F_r &= s_1^r, r, s_2^r, c^r, r', \hat{c}^r; \\ F'_r &= s_2^r, s_1^r, \hat{c}^r, c^r, r, r'; \\ F''_r &= r, r', s_2^r, s_1^r, \hat{c}^r, c^r; \end{aligned} \tag{3}$$

if $N_G(r) = \{s_1, s_2, (s_3)\}$:

$$\begin{aligned} F_r &= s_1^r, r, s_2^r, r', (s_3^r); \\ F'_r &= (s_3^r), s_2^r, s_1^r, r, r'; \\ F''_r &= r, r', (s_3^r), s_2^r, s_1^r. \end{aligned} \tag{4}$$

Third, if room r is adjacent to two or three couples, c_1, c_2 and possibly c_3, and no singles in G, then we set

$$\begin{aligned} F_r &= c_1^r, c_2^r, (c_3^r), r, r', \hat{c}_1^r, \hat{c}_2^r, (\hat{c}_3^r); \\ F'_r &= \hat{c}_1^r, c_1^r, \hat{c}_2^r, c_2^r, (\hat{c}_3^r), (c_3^r), r, r'; \\ F''_r &= r, r', (\hat{c}_3^r), (c_3^r), \hat{c}_2^r, c_2^r, \hat{c}_1^r, c_1^r. \end{aligned} \tag{5}$$

In lists (4) and (5), candidates written within parenthesis may not exist, in which case they should be ignored.

Next, consider a team $p \in S \cup C \cup \hat{C}$. We set the team lists for p depending on the degree of p in G:

if $N_G(p) = \{r_1, r_2, r_3\}$:

$$\begin{aligned} F_p &= p, p', \neg p^{r_1}, \neg p^{r_2}, \neg p^{r_3}; \\ F'_p &= \neg p^{r_3}, p', \neg p^{r_2}, p, \neg p^{r_1}; \\ F''_p &= \neg p^{r_1}, \neg p^{r_2}, \neg p^{r_3}, p, p'. \end{aligned} \tag{6}$$

if $N_G(p) = \{r_1, r_2\}$:

$$\begin{aligned} F_p &= p, p', \neg p^{r_1}, \neg p^{r_2}; \\ F'_p &= \neg p^{r_1}, \neg p^{r_2}, p, p'; \\ F''_p &= p', \neg p^{r_1}, \neg p^{r_2}, p. \end{aligned} \tag{7}$$

Finally, for each dummy team $d \in D$, we let

$$\begin{aligned} F_d &= a_d, b_d, c_d; \\ F'_d &= c_d, a_d, b_d; \\ F''_d &= b_d, c_d, a_d. \end{aligned} \tag{8}$$

This finishes the definition of the team lists F_t, F'_t, and F''_t for each team $t \in T$. Observe that the sets of candidates in F_t taken over each $t \in T$ form a partition of the set X of relevant candidates.

Preferences. In what follows, it will be convenient to fix an ordering over T and use the notation $T = \{t_1, \ldots, t_\rho\}$. Consider the election $\mathcal{E}_q$ introduced in Definition 3.5 over $3^q = \rho$ candidates. Since $|T| = 3^q$, there exists a bijection $\psi : C_q \to T$ between candidates of $\mathcal{E}_q$ and teams in T that maps $t_i \in T$ to the i-th candidate in the preference list of w. Using the alias $\widetilde{t_i} = \psi^{-1}(t_i)$ for each team $t_i \in T$, the election $\mathcal{E}_q$ can be written as

$$\text{election } \mathcal{E}_q : \quad \begin{aligned} w &: \widetilde{t}_1, \widetilde{t}_2, \ldots, \widetilde{t}_\rho; \\ w' &: \widetilde{t}_{\pi(1)}, \widetilde{t}_{\pi(2)}, \ldots, \widetilde{t}_{\pi(\rho)}; \\ w'' &: \widetilde{t}_{\widetilde{\pi}(1)}, \widetilde{t}_{\widetilde{\pi}(1)}, \ldots, \widetilde{t}_{\widetilde{\pi}(\rho)} \end{aligned} \tag{9}$$

for some permutations π and $\widetilde{\pi}$ over $[\rho]$.

We define the permutations φ and $\widetilde{\varphi}$ over $[3\rho]$ based on the election $\mathcal{E}_{q+1}$ similarly: after renaming the candidates in the election $\mathcal{E}_{q+1}$ as $\widetilde{c}_1, \widetilde{c}_2, \ldots, \widetilde{c}_{3\rho}$, the election $\mathcal{E}_{q+1}$ can be re-written as

$$\text{election } \mathcal{E}_{q+1} : \quad \begin{aligned} w &: \widetilde{c}_1, \widetilde{c}_2, \ldots, \widetilde{c}_{3\rho}; \\ w' &: \widetilde{c}_{\varphi(1)}, \widetilde{c}_{\varphi(2)}, \ldots, \widetilde{c}_{\varphi(3\rho)}; \\ w'' &: \widetilde{c}_{\widetilde{\varphi}(1)}, \widetilde{c}_{\widetilde{\varphi}(1)}, \ldots, \widetilde{c}_{\widetilde{\varphi}(3\rho)}; \end{aligned}$$

for some permutations φ and $\widetilde{\varphi}$ over $[3\rho]$.

Now we are ready to give the preferences of voters v, v', and v'':

$$\begin{aligned} v &: F_{t_1}, F_{t_2}, \ldots, F_{t_\rho}, b_1, b_2, \ldots, b_{3\rho-1}, a_1, b_{3\rho}, a_2, a_3, \ldots, a_{3\rho}; \\ v' &: a_{\varphi(1)}, \ldots, a_{\varphi(3\rho)}, F'_{t_{\pi(1)}}, \ldots, F'_{t_{\pi(\rho)}}, b_{\varphi(1)}, \ldots, b_{\varphi(3\rho)}; \\ v'' &: b_{\widetilde{\varphi}(1)}, \ldots, b_{\widetilde{\varphi}(3\rho)}, a_{\widetilde{\varphi}(1)}, \ldots, a_{\widetilde{\varphi}(3\rho)}, F''_{t_{\widetilde{\pi}(1)}}, \ldots, F''_{t_{\widetilde{\pi}(\rho)}}. \end{aligned}$$

Hence, the constructed election is obtained from (9) by substituting each candidate corresponding to some team t_i with the team lists for t_i, and adding the simple candidates in the appropriate manner.

It is clear that the construction takes polynomial time, since building the elections $\mathcal{E}_q$ and $\mathcal{E}_{q+1}$ takes time polynomial in 3^q, and $q = \log_3(|T|)$. Therefore, it remains to prove its correctness.

Connection between solvability of the input instance and flatness of the election restricted to relevant candidates. To prove the correctness of our reduction, let us start with the following facts, which rely on Observation 3.

- Candidate a_1 defeats $b_{3\rho}$, all relevant candidates, and no candidate in $B \setminus \{b_{3\rho}\}$; additionally a_1 also defeats exactly half of the candidates in $A \setminus \{a_1\}$. Therefore,
$$\mathrm{Cpl}_{\mathcal{E}}(a_1) = 1 + 3\rho + \frac{3\rho - 1}{2} = \frac{9\rho + 1}{2} \tag{10}$$
because X is the union of 3ρ parties.[3]
- Candidate $a_i \in A \setminus \{a_1\}$ defeats all relevant candidates, no candidates in B, and half of the candidates in $A \setminus \{a_i\}$. Thus, $\mathrm{Cpl}_{\mathcal{E}}(a_i) = 3\rho + \frac{3\rho-1}{2} = \frac{9\rho-1}{2}$.
- Candidate $b_{3\rho}$ defeats all candidates in A except for a_1, no relevant candidates, and half of the candidates in $B \setminus \{b_{3\rho}\}$. Thus, $\mathrm{Cpl}_{\mathcal{E}}(b_{3\rho}) = |A| - 1 + \frac{3\rho-1}{2} = \frac{9\rho-3}{2}$.
- Candidate $b_i \in B \setminus \{b_{3\rho}\}$ defeats all candidates in A, no relevant candidates, and half of the candidates in $B \setminus \{b_i\}$. Thus, $\mathrm{Cpl}_{\mathcal{E}}(b_i) = |A| + \frac{3\rho-1}{2} = \frac{9\rho-1}{2}$.
- Relevant candidates defeat all candidates in B and no candidates in A.

Due to (10), the above observations imply that a_1 is the unique winner in of the election $\mathcal{E}$ resulting from some nominations if and only if all relevant nominees defeat at most $\frac{3\rho-1}{2}$ relevant nominees, i.e., if the election $\mathcal{E}$ restricted to relevant nominees is flat. In other words, our instance of Possible President is a "yes"-instance if and only if there exist nominations of all parties corresponding to singles, couples, and rooms for which the *relevant election* $\mathcal{E}_X$ below reduced to these nominations becomes flat:

$$\text{relevant election } \mathcal{E}_X: \quad \begin{aligned} v &: F_{t_1}, F_{t_2}, \dots, F_{t_\rho}; \\ v' &: F'_{t_{\pi(1)}}, F'_{t_{\pi(2)}}, \dots, F'_{t_{\pi(\rho)}}; \\ v'' &: F''_{t_{\bar{\pi}(1)}}, F''_{t_{\bar{\pi}(2)}}, \dots, F''_{t_{\bar{\pi}(\rho)}}. \end{aligned} \tag{11}$$

Recall that $\psi : C_q \to T$ is a bijection between candidates of $\mathcal{E}_q$ and teams in T. Comparing (9) and (11), we get the following.

Observation 4. *Replacing each candidate z in the preference lists of w, w', and w'' in the election $\mathcal{E}_q$ with $F_{\psi(z)}$, $F'_{\psi(x)}$, and $F''_{\psi(x)}$, respectively, yields exactly the preference lists of voters v, v', and v'' in the relevant election $\mathcal{E}_X$.*

Observation 4 enables us to take advantage of the structure of election $\mathcal{E}_q$ to establish analogous properties of the constructed instance. Using the specifics of the team list definitions, we can show that our instance of Maximum Matching with Couples admits a complete matching if and only if $\mathcal{E}_X$ admits nominations resulting in a flat election; as we have seen, the latter happens if and only if the constructed instance of Possible President is a "yes"-instance. See the full version [18] for the rest of the proof.

3.3 Four or More Voters

Contrasting Theorem 3.1, showing the tractability of Possible President for Llull with two voters, a reduction from 3-Coloring yields NP-hardness for four voters. As it is possible to add two voters with opposite preferences without changing the election outcome, Theorems 3.2, 3.3 and 3.6 imply Theorem 1.1.

Theorem 3.6 (★). *Possible President for Copeland[1] (i.e., Llull) is* NP*-complete even for four voters and maximum party size $\sigma = 2$.*

3.4 Few Parties

In this section we consider the parameterization of Possible President by t, the number of parties. As we will see, intractability persists even if the number of voters is four, and t is a parameter. Our starting point is Theorem 3.7 which shows that Possible President for Copeland$^\alpha$ for $\alpha < 1$ is W[1]-hard with parameter t.

Theorem 3.7. *For any constant $\alpha \in [0, 1)$, Possible President for Copeland$^\alpha$ is* W[1]*-hard when parameterized by t.*

Proof. We provide a reduction from the Multicolored Clique problem. An instance of this problem consists of a graph $G = (U, E)$ with its vertex set partitioned into k independent sets $U_1, \dots, U_k$, and the question is whether G contains a clique of size k. Multicolored Clique is W[1]-hard when parameterized by k [12, 17].

We construct an instance of Possible President as follows. The set of candidates is $C = U \cup \{p, p'\}$, our distinguished party is $P = \{p\}$, and we have further parties $P' = \{p'\}$ and U_i for each $i \in [k]$. Thus, we have $t = k + 2$ parties.

The set of voters corresponds to the set of "non-edges" in G, that is, to $\bar{E} = \{uu' : u \in U_i, u' \in U_j, i < j, uu' \notin E\}$. Namely, for each $e = uu' \in \bar{E}$, we create two voters v_e and v'_e with preferences as in (12). We fix an arbitrary ordering over C, and write $\overrightarrow{X}$ for listing a set X of candidates according to this order, and $\overleftarrow{X}$ for its reverse.

$$\begin{aligned} v_e &: u, u', \overrightarrow{U \setminus \{u, u'\}}, p, p' \\ v'_e &: p, p', \overleftarrow{U \setminus \{u, u'\}}, u, u' \end{aligned} \tag{12}$$

Consider a reduced election $\mathcal{E}$ obtained by some nominations of all parties. Notice that $\mathrm{Cpl}^\alpha_{\mathcal{E}}(p) = \alpha k + 1$ and $\mathrm{Cpl}^\alpha_{\mathcal{E}}(p') = \alpha k$, since p defeats p', and both are tied with every other candidate.

Assume that G admits a multicolored clique $S = \{u^{(i)}, \dots, u^{(k)}\}$ with $u^{(i)} \in U_i$ for $i \in [k]$. Let each party U_i nominate $u^{(i)}$. As S is a clique in G, it is an independent set in the complement of G, so there is no $e \in \bar{E}$ containing two vertices of S corresponding to two nominated candidates. Thus, each nominee from U obtains a Copeland$^\alpha$ score of $\alpha(k + 1)$. Since $\alpha < 1$, this is strictly smaller than $\mathrm{Cpl}^\alpha_{\mathcal{E}}(p)$, so p is the unique winner of the resulting election.

Conversely, assume for the sake of contradiction that p is the unique winner of some reduced election $\mathcal{E}$, but the nominated candidates in U do not form a clique. Let $u^{(i)} \in U_i$ be a nominee such that there is an edge $e \in \bar{E}$ in the complement of G between $u^{(i)}$ and some nominee $u^{(j)} \in U_j$ with $i < j$; we choose i as the minimal index where this happens. Then, due to the two voters corresponding to $e \in \bar{E}$ we know that candidate $u^{(i)}$ defeats candidate $u^{(j)}$, and due to our choice of i, there is no nominated candidate that defeats $u^{(i)}$. Hence, $\mathrm{Cpl}^\alpha_{\mathcal{E}}(u^{(i)}) \geq \alpha k + 1$, a contradiction to our assumption that p is the unique winner in $\mathcal{E}$. □

We can strengthen Theorem 3.7 as follows:

[3]Henceforth, we write $\mathrm{Cpl}_{\mathcal{E}}(x)$ for the score of candidate x whenever α is irrelevant.

Theorem 3.8 (★). *For any constant $\alpha \in [0,1]$, Possible President for Copeland$^\alpha$ is $\mathsf{W}[1]$-hard when parameterized by t, the number of parties, even if there are only four voters.*

We prove Theorem 3.8 in two steps, first for $\alpha < 1$, and then filling the gap with a more involved reduction for $\alpha = 1$; see the full version [18]. Regarding elections with an odd number of voters, we were able to prove the following:

Theorem 3.9 (★). *For any constant $\alpha \in [0,1]$, Possible President for Copeland$^\alpha$ is $\mathsf{W}[1]$-hard when parameterized by t, the number of parties, even if there are only five voters.*

Each of these results uses a reduction from Multicolored Clique, but the constructions become gradually more complicated; the proof of Theorem 3.9 necessitates also the notion of flat elections (see the full version [18]).

We remark that Possible President is in XP when parameterized by t, assuming that winner determination can be performed in polynomial time: there are at most σ^t possibilities for how parties can choose their nominated candidates, so we can check whether the distinguished party wins in at least one election resulting from some nomination strategy in $\sigma^t n^{O(1)}$ time (see e.g., [19]).

4 MAXIMIN VOTING RULE

Turning to the Maximin voting rule, we investigate how the complexity of Possible President for Maximin depends on the number of voters (Section 4.1) and on the number of parties (Section 4.2).

4.1 Few Voters

We start by extending the tractability result of Theorem 3.1, dealing with the Llull voting rule with two voters, to the Maximin voting rule with two or three voters.

For two voters, tractability again relies on Observation 1 stating the transitivity of the "defeat" relation. For three voters we say that candidate a *strongly defeats* a candidate b, if all three voters prefer a to b. It is easy to see that the "strong defeat" relation is also transitive. This implies that some candidate is the unique winner in a Maximin election if and only if it defeats every other candidate.

Theorem 4.1. *Possible President for the Maximin voting rule is polynomial-time solvable if there are only two voters.*

Proof. The theorem hinges on the fact that a nominee p is a unique winner in a Maximin election $\mathcal{E}$ if and only if p defeats every other nominee. To see this, first realize that if p defeats all nominees then $\mathsf{MM}_\mathcal{E}(p) = 2$ and we have $\mathsf{MM}_\mathcal{E}(c) = 0$ for every other nominee c, so p is the unique winner.

Now assume that p is the unique winner of a reduced election $\mathcal{E}$. Clearly, p cannot be defeated by any nominee, as that would yield $\mathsf{MM}_\mathcal{E}(p) = 0$. Neither is $\mathsf{MM}_\mathcal{E}(p) = 1$ possible, as in this case every other nominee c must have Maximin-score $\mathsf{MM}_\mathcal{E}(c) = 0$, i.e., has to be defeated by at least one other nominee. However, by a similar argument as in the proof of Theorem 3.1, this quickly leads to a contradiction, because the "defeat" relation cannot contain cycles.

Therefore, only $\mathsf{MM}_\mathcal{E}(p) = 2$ is possible, and thus p defeats all nominees. Hence the same quadratic-time algorithm as in Theorem 3.1 solves the Possible President problem also for the Maximin voting rule in the case of two voters. □

Theorem 4.2. *Possible President for the Maximin voting rule is polynomial-time solvable if there are only three voters.*

Proof. Again, we show that p is a unique winner in some election $\mathcal{E}$ if and only if p defeats every other nominee in $\mathcal{E}$. To see this, first realize that if p defeats all nominees, then $\mathsf{MM}_\mathcal{E}(p) \geq 2$ and $\mathsf{MM}_\mathcal{E}(c) \leq 1$ for every other nominee c, so p is the unique winner.

Now assume that p is the unique winner in some election $\mathcal{E}$. Clearly, $\mathsf{MM}_\mathcal{E}(p) = 0$ is impossible. If $\mathsf{MM}_\mathcal{E}(p) = 1$, then every other nominee c must have $\mathsf{MM}_\mathcal{E}(c) = 0$. This means that every nominee other than p is strongly defeated by another nominee. However, this is not possible, as there can be no cycles in the "strong defeat" relation due to its transitivity. Therefore, $\mathsf{MM}_\mathcal{E}(p) \geq 2$ must hold, and thus p can become the unique winner in an election resulting from some nominations if and only if each party has a candidate that is defeated by p. From this, the polynomial-time solvability of the problem follows easily. □

The following theorem shows that for $n \geq 4$ voters, Possible President for Maximin is hard even when all parties have size at most 2. The proof of Theorem 4.3 deals with the case of even and odd number of voters separately, providing two reductions from 3-SAT. Theorems 4.1, 4.2, and 4.3 together prove Theorem 1.2.

Theorem 4.3 (★). *Possible President for Maximin is NP-complete even for instances where the number of voters is a fixed constant $n \geq 4$, and the maximum party size is $\sigma = 2$.*

4.2 Few Parties

Contrasting Theorem 3.7, we show that if the number of parties is small, then we can efficiently solve Possible President for Maximin. More precisely, we provide an FPT algorithm for this problem with parameter t, the number of parties. This subsection is dedicated to proving the following result.

Theorem 4.4 (★). *There exists an algorithm that solves Possible President for Maximin and runs in FPT time with parameter t.*

Let our input instance I of Possible President be an election $\mathcal{E}_0 = (V, C, \{\succ_v\}_{v \in V})$ whose candidate set C is partitioned into a family $\mathcal{P}$ of parties containing a distinguished party $P^\star \in \mathcal{P}$. Our algorithm AlgMM first makes certain guesses about the properties of a hypothetical solution to I, i.e., a nomination strategy that allows $P^\star$ to become the unique winner in the resulting reduced election $\mathcal{E}$. Then, after some preprocessing steps, we reduce our problem to the following directed variant of the Partitioned Subgraph Isomorphism problem [1, 15].

Problem **Partitioned Subdigraph Isomorphism:**
Input: Digraphs D and H with labelling $\gamma : V(H) \to V(D)$.
Question: Is there a subdigraph $\widetilde{H}$ of H that is isomorphic to D, and an isomorphism $f : V(D) \to V(\widetilde{H})$ that maps each vertex v of D to a vertex of $\widetilde{H}$ with label v, i.e., satisfies $\gamma(f(v)) = v$?

Given an instance of Partitioned Subdigraph Isomorphism, we may refer to D and H as the *pattern* and the *host* graphs, respectively. We say that $\widetilde{H}$ is *γ-isomorphic* to D if it satisfies the requirements given in the problem definition.

It is easy to see that Partitioned Subgraph Isomorphism is NP-complete, e.g., by a simple reduction from Multicolored

Clique; see the results by Marx [15] for much stronger lower bounds for the undirected version. However, we will only need to solve Partitioned Subdigraph Isomorphism in the easy special case when all vertices of the pattern graph have indegree at most 1.

Lemma 4.5. *Partitioned Subdigraph Isomorphism can be solved in $O(|V(H)|^2)$ time if the pattern graph D has maximum indegree 1.*

Proof. For a vertex v in a digraph G, let $N_G^-(v)$ and $N_G^+(v)$ denote v's in- and outneighbors in G, respectively. We will also use the notation $\Gamma_v = \{x \in V(H) : \gamma(x) = v\}$ for the set of vertices in H with label v for some $v \in V(D)$.

We introduce two simple rules that reduce the size of the input instance without changing its solvability. The first rule deals with vertices in the pattern graph that have indegree 1 and outdegree 0.

Rule A. Let (D, H, γ) be an instance of Partitioned Subdigraph Isomorphism containing a vertex $v \in V(D)$ with $N_D^+(v) = \emptyset$ and $N_D^-(v) = \{u\}$. Then delete v from D, and delete all vertices of Γ_u without out-neighbors in Γ_v, as well as Γ_v itself from H.

The second rule deals with vertices in the pattern graph that have both in- and outdegree 1.

Rule B. Let (D, H, γ) be an instance of Partitioned Subdigraph Isomorphism containing a vertex $v \in V(D)$ with $N_D^+(v) = \{w\}$ and $N_D^-(v) = \{u\}$ such that (u, w) is not an arc in D. First delete v from D and add the arc (u, w) to D. Second, delete Γ_v from H, and replace the arcs of H contained in $\Gamma_u \times \Gamma_w$ with the arc set

$$A_{uw} = \{(x, y) : x \in \Gamma_u, y \in \Gamma_w, N_H^+(x) \cap N_H^-(y) \cap \Gamma_v \neq \emptyset\}.$$

Claim 1 (★). *Applying Rule A or B yields an equivalent instance of Partitioned Subdigraph Isomorphism.*

Applying Rules A and B preserves the property that all vertices in the pattern graph have indegree at most 1. After applying Rule A exhaustively, we obtain an instance where all vertices of the pattern graph have in- and outdegree at most one, i.e., the pattern graph is a disjoint union of directed cycles, paths, and isolated vertices. In fact, since Rule A is applicable whenever the pattern graph has a connected component that is a directed path with at least two vertices, we know that after the exhaustive application of Rule A we arrive at a pattern graph that is a disjoint union of directed cycles and isolated vertices. Applying then Rule B exhaustively we arrive at an instance $I^\star$ whose pattern graph $D^\star$ consists solely of isolated vertices, possibly with loops. Solving such an instance $I^\star$ is easy: $I^\star$ is a "yes"-instance if and only if the host graph contains a vertex $f(v)$ with label v for each $v \in V(D^\star)$, with $f(v)$ having a loop whenever v has an incident loop in $D^\star$.

Notice that applying either of the two rules consists of the deletion of vertices and, possibly, the addition of arcs to the host graph. Starting from an instance (D, H, γ), the total time spent on the former is $O(|V(H)| + |V(D)|) = O(|V(H)|)$, whereas the total time spent on the latter is at most $O(|V(H)|^2)$, because no arc is added more than once to H. Hence, the total running time is $O(V|H|)^2$. □

We are now ready to describe the steps of AlgMM when run on the instance $(\mathcal{E}_0, \mathcal{P}, P^\star)$; see the full version [18] for its correctness.

Step 1. Guess the candidate p nominated by $P^\star$ in the reduced election $\mathcal{E}$, as well as its Maximin-score $s^\star = \mathrm{MM}_{\mathcal{E}}(p)$ in $\mathcal{E}$.

Step 2. For each party $P \in \mathcal{P} \setminus \{P^\star\}$, guess a party $P' \in \mathcal{P} \setminus \{P\}$ for which the nominees c and c' of P and P' in $\mathcal{E}$, respectively, satisfy $N_{\mathcal{E}}(c, c') < s^\star$. Let $\delta(P)$ denote the guessed party.

Step 3. Delete every candidate $c \in C$ for which $N_{\mathcal{E}_0}(p, c) < s^\star$.

Step 4. For each party $P \in \mathcal{P}$ such that $\delta(P) = P^\star$, delete all candidates $c \in P$ for which $N_{\mathcal{E}_0}(c, p) \geq s^\star$.

Step 5. Let X be the set of candidates deleted in Steps 3 and 4. If there is a party $P \in \mathcal{P} \setminus \{P^\star\}$ with $P \subseteq X$, then return "no."

Step 6. Construct a digraph D whose vertex set is $\mathcal{P} \setminus \{P^\star\}$ and contains an arc (P', P) if and only if $P' = \delta(P)$; hence, each vertex in D has at most one incoming arc.
Construct also a digraph H over $C \setminus X \setminus P^\star$ in which (c', c) is an arc if and only if $N_{\mathcal{E}_0}(c, c') < s^\star$. We set the label $\gamma(c)$ of each candidate c to be the party containing c.

Step 7. Solve Partitioned Subdigraph Isomorphism on instance $J = (D, H, \gamma)$ using the algorithm of Lemma 4.5, and return "yes" if and only if H admits a subdigraph γ-isomorphic to D. Otherwise return "no."

5 CONCLUSIONS AND FUTURE RESEARCH

We provided a detailed multivariate complexity analysis of the Possible President problem in the framework of candidate nomination by parties for several Condorcet-consistent rules; see Table 1 for a summary. Our results show a clear difference between Copeland$^\alpha$ for $\alpha \in [0, 1]$ and Maximin: although both remain NP-hard even for a constant number of voters, Possible President for Maximin becomes tractable (in the parameterized sense) in the realistic scenario where the number t of parties is small, while Copeland$^\alpha$ remains intractable even then. An intriguing question we left open is whether Possible President for Copeland with two or three voters becomes FPT when parameterized by t.

For another promising research direction, recall that our algorithms for two voters relied on the transitivity of the "defeat" relation. Interestingly, the defeat relation is transitive for any number of voters if preferences are single-peaked. Faliszewski et al. [10] proved that Possible President for Plurality remains NP-complete for such preferences. Misra [16] strengthened this result by showing NP-hardness for 1D-Euclidean profiles that are both single-peaked and single-crossing, even with maximum party size 2. What is the situation for voting rules other than Plurality?

The related Necessary President problem, asking if some candidate of a given party can become the winner regardless of nominations from other parties, was shown to be coNP-complete for Plurality by Faliszewski et al. [10], even with maximum party size two. Cechlárová et al. [6] added the analogous results for ℓ-Approval, ℓ-Veto, and Plurality with run-off, and gave integer programs for Necessary President for further voting rules including Copeland, Llull, and Maximin. As far as we know, the parameterized complexity of this problem has not been considered yet.

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REFERENCES

[1] Noga Alon, Raphael Yuster, and Uri Zwick. 1995. Color-coding. *Journal of the ACM* 42, 4 (1995), 844–856. https://doi.org/10.1145/210332.210337

[2] Jørgen Bang-Jensen and Gregory Z. Gutin. 2008. *Digraphs: Theory, Algorithms and Applications* (2nd ed.). Springer, London. https://doi.org/10.1007/978-1-84800-998-1

[3] Péter Biró and Eric McDermid. 2014. Matching with sizes (or scheduling with processing set restrictions). *Discrete Applied Mathematics* 164 (2014), 61–67. https://doi.org/10.1016/j.dam.2011.11.003

[4] Péter Biró and Eric J. McDermid. 2010. *Matching with sizes (or scheduling with processing set restrictions).* Technical Report TR-2010-307. University of Glasgow, Department of Computing Science. Available at https://real.mtak.hu/13519/7/BiroM10tr.pdf.

[5] Allan Borodin, Omer Lev, Nisarg Shah, and Tyrone Strangway. 2024. Primarily about primaries. *Artificial Intelligence* 329 (2024), 104095. https://doi.org/10.1016/j.artint.2024.104095

[6] Katarína Cechlárová, Julien Lesca, Diana Trellová, Martina Hančová, and Jozef Hanč. 2023. Hardness of candidate nomination. *Autonomous Agents and Multi-Agent Systems* 37 (2023), article 37. https://doi.org/10.1007/s10458-023-09622-9

[7] Marek Cygan, Fedor V Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. 2015. *Parameterized algorithms.* Springer, Cham. https://doi.org/10.1007/978-3-319-21275-3

[8] Reinhard Diestel. 2005. *Graph Theory.* Graduate Texts in Mathematics, Vol. 173. Springer-Verlag, Berlin, Heidelberg. https://doi.org/10.1007/978-3-662-53622-3

[9] Rodney G. Downey and Michael R. Fellows. 2013. *Fundamentals of Parameterized Complexity.* Springer, London. https://doi.org/10.1007/978-1-4471-5559-1

[10] Piotr Faliszewski, Laurent Gourvès, Jérôme Lang, Julien Lesca, and Jérôme Monnot. 2016. How hard is it for a party to nominate an election winner?. In *IJCAI 2016: Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence.* AAAI Press, 257–263. https://dl.acm.org/doi/10.5555/3060621.3060658

[11] Piotr Faliszewski, Edith Hemaspaandra, Lane A Hemaspaandra, and Jörg Rothe. 2009. Llull and Copeland voting computationally resist bribery and constructive control. *Journal of Artificial Intelligence Research* 35 (2009), 275–341.

[12] Micheal R. Fellows, Danny Hermelin, Frances A. Rosamond, and Stéphane Vialette. 2009. On the parameterized complexity of multiple-interval graph problems. *Theoretical Computer Science* 410 (2009), pp. 53–61. https://doi.org/10.1016/j.tcs.2008.09.065

[13] Edward B. Foley. 2021. Tournament elections with round-robin primaries: a sports analogy for electoral reform. *Winsonsin Law Review* 1187 (2021), 1187–1229.

[14] Grzegorz Lisowski. 2022. Strategic nominee selection in tournament solutions. In *EUMAS 2022: Proceedings of the 19th European Conference on Multi-Agent Systems.* Springer, 239–256. https://doi.org/10.1007/978-3-031-20614-6_14

[15] Dániel Marx. 2010. Can you beat treewidth? *Theory of Computing* 6 (2010), 85–112. https://doi.org/10.4086/toc.2010.v006a005

[16] Neeldhara Misra. 2019. On the parameterized complexity of party nominations. In *ADT 2019: Proceedings of the 6th International Conference on Algorithmic Decision Theory.* Springer, 112–125. https://doi.org/10.1007/978-3-030-31489-7_8

[17] Krzysztof Pietrzak. 2003. On the parameterized complexity of the fixed alphabet shortest common supersequence and longest common subsequence problems. *Journal of Computer and System Sciences* 67, 4 (2003), 757–771. https://doi.org/10.1016/S0022-0000(03)00078-3

[18] Ildikó Schlotter and Katarína Cechlárová. 2025. Candidate nomination for Condorcet-consistent voting rules. *CoRR* arXiv:2502.03197 [cs.GT] (2025). https://doi.org/10.48550/arXiv.2502.03197

[19] Ildikó Schlotter, Katarína Cechlárová, and Diana Trellová. 2024. Parameterized complexity of candidate nomination for elections based on positional scoring rules. *Autonomous Agents and Multi-Agent Systems* 38 (2024), article 28. https://doi.org/10.1007/s10458-024-09658-5

[20] Markus Schulze. 2011. A new monotonic, clone-independent, reversal symmetric, and condorcet-consistent single-winner election method. *Social Choice and Welfare* 36 (2011), 267–303. https://doi.org/10.1007/s00355-010-0475-4