The Strong Core of Housing Markets with Partial Order Preferences

Ildikó Schlotter
HUN-REN Centre for Economic and Regional Studies
Budapest, Hungary
schlotter.ildiko@krtk.hun-ren.hu

Lydia Mirabel Mendoza-Cadena
MTA-ELTE Matroid Optimization Research Group,
Department of Operations Research, Eötvös Loránd
University
Budapest, Hungary
lyd21@student.elte.hu

ABSTRACT

We study the strong core of housing markets when agents' preferences over houses are expressed as partial orders. We provide a structural characterization of the strong core, and propose an efficient and weakly group-strategyproof algorithm that finds an allocation in the strong core or decides that it is empty, even in the presence of forced and forbidden arcs. Additionally, we show that certain results known for the strong core in the case when agents' preferences are weak orders can be extended to the setting with partial order preferences; among others, we show that the strong core in such housing markets satisfies the property of respecting improvements.

KEYWORDS

housing market; strong core; partial orders; forced and forbidden arcs; strategyproofness; property of respecting improvement

1 INTRODUCTION

A Shapley-Scarf housing market involves a set of agents, each endowed with exactly one unit of some indivisible good – a *house* – and having preferences over the houses owned by other agents in the market. Agents are assumed to trade among themselves without monetary transfers; hence, the outcome in such a market is an *allocation* that assigns to each agent exactly one house, possibly its own. Housing markets have been extensively studied in the fields of economics and computer science since the seminal work of Shapley and Scarf in 1974 [43]. Their most prominent motivation comes from kidney-exchange programs [10, 12, 37], but applications include various exchange markets[1] ranging from time-sharing markets [45] and time banks [3] to on-campus housing [1].

Two prominent solution concepts widely investigated in connection to housing markets are the *core* and the *strong core*: the *core* of

a housing market contains allocations from which no coalition of agents can deviate so that each of them strictly improves their situation by trading among themselves, while the *strong core* (or *strict core*) contains allocations that do not admit a coalition of agents who are able to weakly improve their situation, with at least one agent strictly improving as a result of the deviation. Shapley and Scarf described the Top Trading Cycle (TTC) mechanism, attributed to David Gale, that always finds an allocation in the core [43]. Roth and Postlewaite showed that if agents' preferences are strict, then the TTC mechanism always returns the unique allocation in the strong core [36]. Roth proved that in such a model the TTC is *strategyproof*, meaning that no agent can improve its outcome under TTC by misreporting its preferences [35]. Later, TTC under strict preferences was shown to be even *group-strategyproof* [9, 40], meaning that no set of agents can improve their situation by misreporting their preferences in a coordinated fashion.

If agents can be indifferent between houses, then both the core and the strict core loose many of their appealing properties. Although the TTC mechanism always returns an allocation in the core even when agents' preferences over the houses are *weak orders* (i.e., linear orders containing ties) [43], the core in such markets may contain multiple allocations. As opposed to the core, the strong core can become empty if agents' preferences are weak orders. Quint and Wako characterized the strong core in such housing markets, and gave a polynomial-time algorithm that either finds an allocation in the strong core or concludes that the strong core is empty [33]. Several researchers have introduced generalizations of the TTC in search of a mechanism that maintains the desirable properties of TTC even under the presence of indifference in the market; this line of research has yielded algorithms that efficiently find allocations in the core with additional properties such as Pareto-optimality, while ensuring also strategyproofness [2, 5, 25, 32, 38].

Despite the significant effort to deal with weakly ordered preferences in housing markets, there is still limited knowledge about the case when agents' preferences can be *partial orders*, a generalization of weak orders. Partial orders arise in contexts where agents may regard two alternatives as *incomparable*; as opposed to indifference in weak orders, incomparability is not necessarily transitive. Despite the clear motivations for studying partial orders in the context of kidney exchange, as we explain it below, there is scant literature on housing markets that allow agents to express their preferences using partial orders. A recent study [41] addresses questions about the core in such a model, but we are not aware of any research concerning the strong core in a setting where preferences can be partial orders. This paper aims to address this gap in the literature.

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[1]For an online exchange market for board games, see https://boardgamegeek.com/wiki/page/Math_Trades.

Motivation. Partial orders naturally arise in real-life applications where agents compare alternatives based on multiple unrelated criteria. An important example of this occurs in kidney exchange, where patients exchange the incompatible organs donated by their own donors; such a situation can be modelled as a housing market where agents are incompatible patient–donor pairs. The preferences of patients over possible kidney transplants are primarily based on two factors: the age of the donor and the HLA compatibility between donor and recipient. In fact, in the UK kidney exchange program, these two factors are the only ones for which patients can set a threshold of acceptance [12]. The unrelated nature of these two factors gives rise to a partial ordering: we can only assert that a certain graft is better than another if it is more advantageous regarding both of these factors, so a graft from a younger donor but with a worst HLA-matching is incomparable with a graft from an older donor but with a better HLA-matching. Such incomparability is not necessarily transitive; hence, patients' preferences over possible transplants is only a partial, and not a weak, ordering.

Another source of partial order preferences in kidney exchange (or, in fact, in many other applications) is the fact that small differences between two alternatives may not be significant enough to influence the preferences, but slight differences may add up to a more substantial contrast. For example, when the age difference between two possible donors is, say, less than a year, then a recipient might regard the corresponding grafts as equivalent; however, such incomparability is not transitive: donors A and B, as well as B and C, might have an age difference less than a year, but A might be almost two years older than C. Such preferences can be described as so-called *semi-orders*, a special case of partial orders.

Another application where partial orders arise naturally is a project aimed at promoting social inclusion of individuals with intellectual disabilities, where participants are intellectually disabled people who have to be assigned to various job positions, namely, to computer-numerical-controlled (CNC) machines in a facility [46].[2] The preferences of the participants over the available positions are determined by multiple criteria, based on their abilities and the requirements of the job, and thus induce a partial ordering [13]. See [19] for further applications where partial orders appear.

Our contribution and organization. We consider the strong core of housing markets where agents' preferences are expressed as partial orders, which is a setting that has not been studied before.

After providing the preliminaries in Section 2, we explain why the Quint–Wako characterization fails when agents' preferences are partial—and not weak—orders (Section 3.1), and present an alternative characterization of the strong core in such a setting (Section 3.2). Even though our characterization does not immediately yield an algorithm for finding an allocation in the strong core (if exists), in Section 4 we propose a polynomial-time algorithm for this problem. Recall that the Quint–Wako algorithm [33] finds an allocation in the strong core when agents' preferences are weak orders, or detects that the strong core is empty; hence, our algorithm can be thought of as a generalization of this result to the setting with partial order preferences. In fact, our algorithm can handle forbidden and forced

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[2]Strictly speaking, this application does not directly concern housing markets; however, assuming a dynamic setting where certain participants are already assigned to job positions while new participants may also enter the market, we quickly arrive at a situation that can be modelled as a housing market.

Table 1: Summary of results on the problem of finding allocations in the core and strong of housing markets. The phrase "f/f arcs" refers to problem variants with forced or forbidden arcs. "NP-h" stands for NP-hardness. If the problem is polynomial-time solvable, we write either "in P" or the name of the corresponding algorithm.

Problem	Strict orders	Weak orders	Partial orders
Core	TTC [43]	TTC [43]	TTC [41, 43]
Core, f/f arcs	NP-h [41]	NP-h [41]	NP-h [41]
Strong core	TTC [36]	Quint–Wako [33]	**Thm 4.4**
Strong core, f/f arcs	TTC [36]	in P [41]	**Thm 4.4**

arcs in the graph underlying the market: it can find an allocation in the strong core that contains all forced arcs but no forbidden arcs, if such an allocation exists; see Sections 4.1–4.3. We prove that our algorithm is (weakly) group-strategyproof (Section 4.4), and we also examine its ability to enumerate all allocations in the strong core (Section 4.5).

In Section 5, we investigate certain properties of the strong core. The main result of Section 5 is that the strong core satisfies the property of *respecting improvement*: assuming that the outcome of a market is an allocation in the strong core, we show that an agent can only (weakly) benefit from an increase in the "value" of its house, i.e., from its house becoming more preferred according to the remaining agents; this observation is a generalization of the analogous result for weakly ordered preferences by Biró et al. [11].

Section 6 contains our conclusions.

We defer some of our results and proofs to the full version of our paper [42]; the symbol ⋆ marks statements with deferred proofs.

Related work. We have already mentioned in the introduction the series of papers containing prior work on the core and the strong core of housing markets, a line of research into which our paper fits smoothly. The closest works to our study are the papers by Quint and Wako [33] and by Schlotter et al. [41]. See Table 1 for a comparison of the results in these works.

Besides the literature on housing markets, our paper also builds on various ideas that have been explored in the broader context of matching under preferences.

First, we emphasize that even though we are not aware of any previous work on housing markets with partial order preferences with the exception of the work by Schlotter et al. on the core [41], partial order preferences have been widely studied in the area of matching under preferences, appearing already in early works on stable matchings [20, 21, 24, 30, 34].

Second, problems involving edge restrictions have also been extensively studied for stable matching problems, see e.g. [16–18, 21, 27]. In the context of housing markets, the question whether a given agent can obtain a certain house (or can avoid ending up with a certain house) has been addressed in [41]. The analogous problems have also been studied in the house allocation model where a set of objects needs to be allocated to a set of agents (without initial endowments) using serial dictatorship [4, 6, 7, 14, 39].

Third, questions about the property of respecting improvement have been already studied in the context of housing markets [37, 41], but this line of research has its roots in a study by Balinski and

Sönmez [8] on college admission. The authors investigated how an improvement in a student's test scores may have undesirable effects. Their results inspired further studies of the same flavor about school choice and other allocation problems [22, 26, 44].

Fourth, the topic of (group-)strategyproofness has a vast literature in the area of social choice, and has been explored in connection to the TTC mechanism and its generalizations [2, 9, 25, 28, 32, 35, 38]; more recent results investigate strategyproofness in relation to different extensions of housing markets [23, 47], and the interesting topic of *obvious strategyproofness* in connection to TTC [29, 31].

2 PRELIMINARIES

For a positive integer $\ell \in \mathbb{N}$, we use the notation $[\ell] = \{1, 2, \ldots, \ell\}$.

Directed graphs. We assume the reader is familiar with basic concepts from graph theory; see the full version for a detailed description of all the definitions we rely on. Given a vertex v in a directed graph $D = (V, E)$, we let $\delta^{in}(v) = \{(u, v) : (u, v) \in E\}$ and $\delta^{out}(v) = \{(v, u) : (v, u) \in E\}$ denote the set of *incoming* and *outgoing* arcs of v, respectively. Given a set $F \subseteq E$ of arcs, we extend this notation by letting $\delta^{in}_F(v) = \delta^{in}(v) \cap F$ and $\delta^{out}_F(v) = \delta^{out}(v) \cap F$. Given a vertex set $V' \subseteq V$ and an arc set $E' \subseteq E$, we let $D[V']$ denote the subgraph *induced* by V' and $D[E']$ for the subgraph *spanned* by E'. A notion of central importance for our purposes is the following: we call a set $S \subseteq V$ an *absorbing set* in D, if S is a strongly connected component of D that is not left by any arc. In other words, when contracting S, the newly introduced vertex corresponding to S has no outgoing arcs.

Partial orders. A partial ordering over a set S is a binary relation $\succ$ over S that is irreflexive (i.e., $s \nsucc s$ for all $s \in S$), antisymmetric (i.e., if $s \succ r$, then $r \nsucc s$), and transitive (i.e., $s \succ r$ and $r \succ q$ implies $s \succ q$). Given a partial ordering $\succ$ over S, two elements s and r of S are *comparable* if $s \succ r$ or $r \succ s$; otherwise, they are *incomparable*, denoted by $s \sim r$. We write $s \succeq r$ for $r \nsucc s$.

Partial orders are a generalization of weak orders. A partial ordering over S is a *weak order* if and only if the incomparability relation $\sim$ is an equivalence relation: it is reflexive, symmetric, and transitive. Note that the first two conditions always hold, hence the transitivity of the incomparability relation is the key property for a partial order to be a weak order. For example, $S = \{s, r, q\}$ with $s \succ q$ but $s \sim r$ and $r \sim q$ is not a weak order, because $s \nsim q$.

Housing markets. A *housing market* is a pair $(N, \{\succ_a : a \in N\})$ where N is the set of *agents* and $\succ_a$ is a partial order over N for each $a \in N$, representing the preferences of agent a. Note that we identify each house with its initial owner and thus express agents' preferences over houses as partial orders over N. Recall that we write $b \sim_a c$ if $b \nsucc_a c$ and $c \nsucc_a b$ both hold, and $b \succeq_a c$ means that $c \nsucc_a b$. We say that *a prefers c to b* if $c \succ_a b$, and *a weakly prefers c* to b, if $c \succeq_a b$. Agent b is *acceptable* to agent a, if $b \succeq_a a$, and we let $A(a) = \{b \in N : b \succeq_a a\}$ denote the *acceptability set* of a.

The underlying graph of a housing market $H = (N, \{\succ_a : a \in N\})$ is defined as the directed graph $D^H = (N, E)$ where vertices correspond to agents, and arcs correspond to acceptability, that is, $E = \{(a, b) : a \in N, b \in A(a)\}$. Note that $(a, a) \in E$ for each $a \in N$, so D^H contains a loop at each agent. We say that an arc $(a, b) \in E$ is *dominated* by an arc (a', b') if $a = a'$ and $b' \succ_a b$. An arc $(a, b) \in E$ is *undominated*, if no arc in E dominates it.

Given a subset $N' \subseteq N$ of agents, the *submarket* of H restricted to the agent set N' is the housing market $H_{|N'} = (N', \{\succ'_a : a \in N'\})$ where $\succ'_a$ is the restriction of $\succ_a$ to the set N' of agents.

An *allocation* in H is defined as a set $X \subseteq E$ of arcs in D^H such that each agent has exactly one outgoing and exactly one incoming arc in X, i.e., $|\delta^{in}_X(a)| = |\delta^{out}_X(a)| = 1$ for each $a \in N$.

Remark 1. Given an arbitrary subset E' of arcs in D^H, it is possible to check in polynomial time whether H admits an allocation containing only arcs of E', by reducing this problem via a well-known, simple reduction to finding a perfect matching in a bipartite graph. This classic reduction constructs an auxiliary graph $G = (N \cup N', F)$ where $N' = \{a' : a \in N\}$ is a copy of the set N, and the edge set is $F = \{ab' : (a, b) \in E'\}$; then allocations in H contained in E' correspond bijectively to perfect matchings in G.

For an allocation X, or more generally, a set X of arcs in D^H such that $|\delta^{out}_X(a)| \leq 1$ for each agent $a \in N$, we let $X(a)$ denote the agent b if $\delta^{out}_X(a) = \{(a, b)\}$; if $\delta^{out}_X(a) = \emptyset$, then we set $X(a) = \varnothing$. We extend the agents' preferences over N to preferences over allocations in the straightforward way: agent a (weakly) prefers an allocation X to an allocation X' if and only if a (weakly) prefers $X(a)$ to $X'(a)$. Extending the notation, we write $X \succ_a X'$ for a preferring X to X'; we use $X \succeq_a X'$ and $X \sim_a X'$ analogously.

An allocation X is in the *strong core* of H, if there does not exist a *blocking cycle* for X, i.e., a cycle C in D^H such that all agents a on C weakly prefer $C(a)$ to $X(a)$, and at least one agent $a^\star$ on C (strictly) prefers $C(a^\star)$ to $X(a^\star)$. We remark that such cycles are often called *weakly* blocking cycles. We adopt the notation by Biró et al. [11] and write $\mathcal{SC}(H)$ for the strong core of H.

An allocation X is in the *core* of H, if there does not exist a *strictly blocking cycle* for X, i.e., a cycle C in D^H such that all agents a on C strictly prefer $C(a)$ to $X(a)$.

3 CHARACTERIZING THE STRONG CORE

If agents' preferences are strict orders, then the strong core always contains a unique allocation [36], which can be found in linear time using the Top Trading Cycle algorithm, attributed to David Gale in the seminal paper by Shapley and Scarf [43]. If preferences are weak orders, then the strong core can be empty, but Quint and Wako provided a characterization of allocations in the strong core in such a setting, and showed how to use this characterization to find an allocation in the strong core if there exists one [33].

In Section 3.1 we show that the characterization of the strong core by Quint and Wako fails to hold in the setting when agents' preferences can be partial orders. In Section 3.2 we generalize the results of Quint and Wako to the setting with partial order preferences by giving a characterization of the strong core.

3.1 The Quint–Wako Characterization Fails for Partial Orders

For housing markets where agents' preferences are weak orders, Quint and Wako proved the following characterization of allocations in the strong core [33]. Recall that an absorbing set is a strongly connected component from which no arc may leave.

THEOREM 3.1 (**Quint–Wako characterization [33]**). *Suppose that $H = (N, \{\succ_a : a \in N\})$ is a housing market where each $\succ_a$ is a*

weak order. Let U *be the set of undominated arcs in* D^H, *and* S *an absorbing set in* $D^H[U]$. *An allocation* X *for* H *is in the strong core of* H *if and only if it can be partitioned into sets* X_S *and* $X_{N\setminus S}$ *where*

   (a) $X_S = X \cap (S \times S)$ *is an allocation in* $H_{|S}$ *contained in* U, *and*

   (b) $X_{N\setminus S} = X \cap (N \setminus S) \times (N \setminus S)$ *is an allocation in the strong core of* $H_{|(N\setminus S)}$.

It is not hard to show that conditions (a) and (b) stated in Theorem 3.1 are *necessary* for a given allocation to be in the strong core also in the case when agents' preferences are partial orders (see the full version for details). Example 3.2 shows that conditions (a) and (b) of Theorem 3.1 are *not sufficient* to ensure that an allocation belongs to the strong core. Thus, the characterization established by Theorem 3.1 for weakly ordered preferences fails to hold in the case when agents' preferences are partial orders.

Example 3.2. Consider the following market H^1 over agent set $N = \{a, b, c, d\}$; see its underlying graph in Figure 1. Let the acceptability sets be defined as $A(a) = A(b) = N$, $A(c) = \{a, b, c\}$, and $A(d) = \{c, d\}$. For each agent x and $y \in A(x) \setminus \{x\}$, we let $y >_x x$; additionally, $c >_a d$ and $c >_b d$.
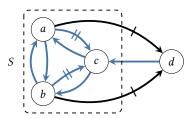

Figure 1: The underlying graph of housing market H^1 **defined in Example 3.2. Henceforth, loops are omitted, and undominated arcs are shown in blue, the strongly connected components they form are shown as dashed polygons. Single and double line markings | and || convey domination: an arc marked with || dominates an arc marked with | and leaving the same agent; e.g.,** (a, c) **dominates** (a, d) **but not** (a, b), **and neither does** (a, b) **dominate** (a, d).

Note that $U = \{(a, b), (a, c), (b, c), (b, a), (c, a), (c, b), (d, c)\}$ is the set of undominated arcs, and the unique absorbing set in $D^{H^1}[U]$ is $S = \{a, b, c\}$. The submarket $H^1_{|S}$ admits two allocations in U: $X_1 = \{(a, b), (b, c), (c, a)\}$ and $X_2 = \{(a, c), (c, b), (b, a)\}$. The only allocation for the submarket $H^1_{|(N\setminus S)} = H_{|\{d\}}$ is $X_d = \{(d, d)\}$.

Although both X_1 and X_2 are in the strong core of $H^1_{|S}$, and X_d is in the strong core of $H_{|N\setminus S}$, neither $X_1 \cup X_d$ nor $X_2 \cup X_d$ is in the strong core of H^1: $\{(a, d), (d, c), (c, a)\}$ is a blocking cycle for $X_1 \cup X_d$, and similarly, $\{(b, d), (d, c), (c, b)\}$ is a blocking cycle for $X_2 \cup X_d$. This shows that Theorem 3.1 does not extend to partial order preferences.

3.2 A Characterization Through Peak Sets

Given an allocation X in H, let us define the following arc sets:

$$E_{[\sim X]} = \{(a, b) \in E : X(a) \sim_a b\},$$
$$E_{[> X]} = \{(a, b) \in E : b >_a X(a)\}, \quad \text{and}$$
$$E_{[\geq X]} = E_{[\sim X]} \cup E_{[> X]}.$$

Let $D_{[\sim X]} = (N, E_{[\sim X]})$, and let the digraphs $D_{[> X]}$ and $D_{[\geq X]}$ be defined analogously. The following definition captures the central notion we need for our characterization of the strong core.

Definition 3.3. Given an allocation X in a housing market H, a *peak set* of X is an absorbing set in $D_{[\geq X]}$.

In our characterization of the strong core, peak sets will play the role that absorbing sets in the subgraph of undominated arcs have in Theorem 3.1. The following lemma states their key property.

LEMMA 3.4 ($\star$). *Suppose that* X *is an allocation in the strong core of* H. *If* S *a peak set of* X, *then* $X \cap (S \times S)$ *is an allocation in* $H_{|S}$ *consisting only of undominated arcs. Moreover, each arc leaving* S *is dominated by an arc of* X.

Based on Lemma 3.4, we can now characterize the strong core.

THEOREM 3.5. *Suppose that* $H = (N, \{>_a : a \in N\})$ *is a housing market where each* $>_a$ *is a partial order, and let* U *be the set of undominated arcs in* D^H. *Then allocation* X *for* H *is in the strong core of* H *if and only if it can be partitioned into two sets* X_1 *and* X_2 *such that there exists a set* $S \subseteq N$ *of agents for which*

   (a) X_1 *is an allocation in* $H_{|S}$ *contained in* U,

   (b) X_2 *is an allocation in the strong core of* $H_{|(N\setminus S)}$, *and*

   (c) *each arc of* D^H *leaving* S *is dominated by an arc of* X_1.

Moreover, S *can be chosen as a peak set for* X.

PROOF. First suppose that X is in the strong core of H. Define S as a peak set of X, and set $X_1 := X \cap (S \times S)$ and $X_2 = X \setminus X_1$. By Lemma 3.4 we know that conditions (a) and (c) are fulfilled. Since X_1 is an allocation on $H_{|S}$, we necessarily have that $X_2 = X \setminus X_1$ is an allocation in $H_{|(N\setminus S)}$. To see that X_2 is in the strong core of this submarket, it suffices to note that any blocking cycle for X_2 in $H_{|(N\setminus S)}$ would also block X in H.

Second, suppose that some allocation X can be partitioned into X_1 and X_2 satisfying conditions (a)–(c) for some set S of agents; we show that X is in the strong core of H. Assume for the sake of contradiction that there is a blocking cycle C for X. First, since X_1 is an allocation for $H_{|S}$ and $X_1 \subseteq U$, we know that C must involve at least one agent not in S. Second, since X_2 is an allocation in the strong core of $H_{|(N\setminus S)}$, we know that C must also contain at least one agent not in $N \setminus S$. Thus, C involves agents from both S and $N \setminus S$, and consequently has to contain at least one arc leaving S. However, all arcs leaving S are dominated by an arc of X_1, which contradicts the assumption that C blocks X. This proves that no blocking cycle for X exists, so X is indeed in the strong core of H. □

4 FINDING AN ALLOCATION IN THE STRONG CORE

In this section, we show how Theorem 3.5 can be used to construct an algorithm that either finds an allocation in the strong core of a housing market where agents' preferences are partial orders, or concludes that such an allocation does not exist.

Before explaining our ideas, let us briefly describe the algorithm by Quint and Wako [33] that solves this problem in the setting where agents' preferences are weak orders, based on Theorem 3.1. Given a housing market $H = (N, \{>_a : a \in N\})$, the Quint–Wako algorithm first computes an absorbing set S in the subgraph $D^H[U]$

of undominated arcs in the underlying digraph. Then it checks whether $H_{|S}$ admits an allocation consisting only of undominated arcs, and if so, stores such an allocation X_S and proceeds with the submarket $H_{|(N\setminus S)}$ of the remaining agents recursively. If the recursive call returns an allocation $X_{N\setminus S}$ in the strong core of $H_{|(N\setminus S)}$, then the algorithm outputs $X_S \cup X_{N\setminus S}$; otherwise it concludes that the strong core of H is empty.

Unfortunately, the characterization of the strong core given in Theorem 3.5 does not readily offer an algorithm for finding an allocation in the strong core of a given housing market, or to decide whether such an allocation exists. The reason for this is that it is not immediately clear how to find a set S of agents for which conditions (a)–(c) of Theorem 3.5 hold for the desired allocation X (or, more precisely, for the arcs sets X_1 and X_2 that together yield X). By Theorem 3.5 we know that the set S can be chosen to be a peak set of the desired allocation; however, this insight does not seem to offer a direct way to find such a set without knowing X itself.

In this section, we propose a method that resolves this issue and finds an allocation in the strong core of the given housing market, if such an allocation exists. In fact, we are going to solve the following more general computational problem:

STRONG CORE WITH FORBIDDEN ARCS (SCFA):

Input: A housing market H with partial orders and a set F of *forbidden* arcs in the underlying directed graph D^H.

Task: Decide if there exists an allocation X in the strong core of H such that $X \cap F = \emptyset$, and if so, find such an allocation.

In Section 4.1 we describe a polynomial-time algorithm to solve the STRONG CORE WITH FORBIDDEN ARCS problem, and we sketch its proof of correctness in Section 4.2. We extend our algorithm to a setting with forced arcs in Section 4.3, and explain its weak group-strategyproofness in Section 4.4. In Section 4.5 we briefly compare it with the Quint–Wako algorithm [33].

4.1 Algorithm for SCFA

We now present Algorithm SCFA to solve our instance (H, F) of SCFA. An allocation in $\mathcal{SC}(H)$ that is disjoint from F is a *solution*.

Description of Algorithm SCFA. The main idea of our algorithm is to find a set $T^\star$ that can serve as the peak set of our desired allocation in the strong core, find an appropriate allocation on $H_{|T^\star}$, and solve the remainder recursively after deleting all agents in $T^\star$.

The algorithm starts by dealing with the case when the market only contains a single agent. Next, it computes the family $\mathcal{S}$ of strongly connected components in the digraph $D^H[U] = (N, U)$ of undominated arcs and, using the method of Remark 1, checks for each component $S \in \mathcal{S}$ whether there is an allocation for the submarket $H_{|S}$ consisting only of undominated non-forbidden arcs. Those sets that pass this test are collected in a family $\mathcal{T} \subseteq \mathcal{S}$.

Given this collection $\mathcal{T} \subseteq \mathcal{S}$, an iterative process is started. At each step in this iteration, Algorithm SCFA checks whether each submarket $H_{|S}$ for $S \in \mathcal{T}$ admits an allocation that contains only arcs that (i) are undominated and not forbidden, and (ii) dominate all arcs leaving $T^\star = \bigcup_{T \in \mathcal{T}} T$; the set of such arcs in denoted by $E_{S,\mathcal{T}}$ on line 13. More formally, let us say that an allocation X in $H_{|S}$ is $T^\star$-*valid* for some $T^\star \subseteq N$, if

(i) $X \subseteq U \setminus F$ and

Algorithm SCFA Solving SCFA.

Input: An instance (H, F) of SCFA where $D^H = (N, E)$.
Output: An allocation in the strong core of H disjoint from F, or $\emptyset$ if no such allocation exists.

1: **if** $|N| = 1$ **then**
2:      **if** $(a, a) \in E \setminus F$ where $N = \{a\}$ **then return** $\{(a, a)\}$.
3:      **else return** $\emptyset$.
4: Let U denote the set of undominated arcs in D^H.
5: Let $\mathcal{S}$ be the set of strongly connected components in (N, U).
6: Let $\mathcal{T} = \emptyset$.
7: **for all** $S \in \mathcal{S}$ **do**
8:      Let $E_S = \{(a, b) \in U \setminus F : a, b \in S\}$.
9:      **if** $\exists$ an allocation for $H_{|S}$ in E_S **then** Put S into $\mathcal{T}$.
10: **while** $|\mathcal{T}| > 0$ **do**
11:      Let $\mathcal{R} = \emptyset$ and $T^\star = \bigcup_{S \in \mathcal{T}} S$.
12:      **for all** $S \in \mathcal{T}$ **do**
13:          Let $E_{S,\mathcal{T}} = \{(a, b) \in E_S : b >_a b' \ \forall (a, b') \in E, b' \notin T^\star\}$.
14:          **if** $\exists$ an allocation for $H_{|S}$ in $E_{S,\mathcal{T}}$ **then**
15:              Let X_S be such an allocation.
16:          **else** Add S as an element to $\mathcal{R}$.
17:      **if** $\mathcal{R} = \emptyset$ **then**
18:          **if** $N = T^\star$ **then return** $\bigcup_{S \in \mathcal{T}} X_S$.
19:          Let $N' = N \setminus T^\star$ and $F' = F \cap (N' \times N')$.
20:          Call $X' = \text{SCFA}(H_{|N'}, F')$.
21:          **if** $X' = \emptyset$ **then return** $\emptyset$
22:          **else return** $\bigcup_{S \in \mathcal{T}} X_S \cup X'$.
23:      **else** Delete each $S \in \mathcal{R}$ from $\mathcal{T}$.
24: **return** $\emptyset$.

(ii) $X(a) >_a b'$ for each $(a, b') \in E$ where $a \in T^\star$ and $b' \notin T^\star$.

Thus, an allocation X in $H_{|S}$ is $T^\star$-valid if and only if

$$X \subseteq \left\{ (a, b) \in U \setminus F : b >_a b' \text{ for each } (a, b') \in E, b' \notin T^\star \right\}.$$

If Algorithm SCFA finds that all submarkets $H_{|S}$, $S \in \mathcal{T}$, admit a $T^\star = \bigcup_{T \in \mathcal{T}} T$-valid allocation, then it stores such an allocation for each $S \in \mathcal{T}$, deletes the agents in $T^\star$, and proceeds with the remaining market recursively—unless the remaining market is empty, in which case no recursion is necessary. If, on the contrary, $H_{|S}$ does not admit a $T^\star$-valid allocation for certain sets $S \in \mathcal{T}$, then each such set S is removed from the collection $\mathcal{T}$.

This iterative process stops either if all sets in $\mathcal{T}$ pass the test of admitting a $\bigcup_{T \in \mathcal{T}} T$-valid allocation, or if $\mathcal{T}$ becomes empty, in which case the algorithm concludes that there is no solution.

Example 4.1. Consider again the housing market H^1 defined in Example 3.2 and depicted in Figure 1, to see how Algorithm SCFA runs with H^1 as its input. For simplicity, we define the set of forbidden arcs as empty. The set of strongly connected components in the subgraph spanned by all undominated arcs is $\mathcal{S} = \{S_1, S_2\}$ where $S_1 = \{a, b, c\}$ and $S_2 = \{d\}$. Algorithm SCFA computes the arc sets $E_{S_1} = \{(a, b), (b, c), (c, a), (a, c), (c, b), (b, a)\}$ and $E_{S_2} = \emptyset$, and finds that $H^1_{|S_1}$ admits an allocation in E_{S_1}, e.g., the allocation $\{(a, b), (b, c), (c, a)\}$, but no allocation for $H^1_{|S_2}$ (containing only the vertex d) can be constructed from the empty set. Thus, Algorithm SCFA initializes the family $\mathcal{T}$ by setting $\mathcal{T} = \{S_1\}$.

Starting the iteration on lines 10–23, the algorithm first computes the arc set $E_{S_1,\mathcal{T}}$, obtaining $E_{S_1,\mathcal{T}} = \{(a,c), (c,a), (b,c), (c,b)\}$; note that neither (a,b) nor (b,a) is contained in $E_{S_1,\mathcal{T}}$, since they do not dominate the arc (a,d) or (b,d), respectively (observe that both (a,d) and (b,d) leave the set of vertices contained in $\mathcal{T}$, that is, $T^\star = S_1 = \{a,b,c\}$). The algorithm next observes that $E_{S_1,\mathcal{T}}$ does not contain an allocation for $H_{|S_1}$; thus, it removes S_1 from $\mathcal{T}$, leaving $\mathcal{T}$ empty. Hence, the iteration stops, and Algorithm SCFA outputs $\varnothing$ on line 24, concluding that the strong core is empty.
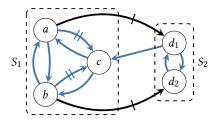

Figure 2: The housing market H^2 defined in Example 4.2.

Example 4.2. Consider the following market H^2 over agent set $N = \{a,b,c,d_1,d_2\}$; see its underlying graph (without loops) on Figure 2. Let the acceptability sets be defined as $A(a) = N \setminus \{d_2\}$, $A(b) = N \setminus \{d_1\}$, $A(c) = \{a,b,c\}$, $A(d_1) = \{c,d_1,d_2\}$, and $A(d_2) = \{d_1,d_2\}$. For each agent $x \in N$ and $y \in A(x) \setminus \{x\}$, we let $y >_x x$; additionally, $c >_a d_1$ and $c >_b d_2$.

Consider how Algorithm SCFA runs with H^2 as its input, without forbidden arcs. All arcs are undominated except for (a,d_1) and (b,d_2). The strongly connected components in the subgraph spanned by all undominated arcs are $S_1 = \{a,b,c\}$ and $S_2 = \{d_1,d_2\}$. Algorithm SCFA computes the arc sets $E_{S_1} = \{(a,b), (b,c), (c,a), (a,c), (c,b), (b,a)\}$ and $E_{S_2} = \{(d_1,d_2), (d_2,d_1)\}$, and finds that both $H^2_{|S_1}$ and $H^2_{|S_2}$ admit an allocation in E_{S_1} and in E_{S_2}, respectively. Thus, Algorithm SCFA initializes $\mathcal{T}$ by setting $\mathcal{T} = \{S_1, S_2\}$.

Starting the iteration on lines 10–23, the algorithm first computes the arc sets $E_{S,\mathcal{T}}$ for both $S \in \mathcal{T}$, obtaining $E_{S_1,\mathcal{T}} = E_{S_1}$ and $E_{S_2,\mathcal{T}} = E_{S_2}$ and storing allocations X_{S_1} and X_{S_2} for $H_{|S_1}$ and for $H_{|S_2}$, respectively. Suppose that the algorithm finds $X_{S_1} = \{(a,b), (b,c), (c,a)\}$ and $X_{S_2} = \{(d_1,d_2), (d_2,d_1)\}$; the case when $X_{S_1} = \{(a,c), (c,b), (b,a)\}$ is similar. The algorithm finds on line 18 that $N = T^\star = S_1 \cup S_2$, and hence returns the allocation $X_{S_1} \cup X_{S_2}$.

4.2 Correctness of Algorithm SCFA

The correctness of Algorithm SCFA relies on the following key fact.

LEMMA 4.3. *If X is an allocation in the strong core of H disjoint from F, and P is a peak set of X, then $P \subseteq \bigcup_{T \in \mathcal{T}} T$ always holds on lines 10–23 during the execution of Algorithm SCFA on (H,F).*

PROOF. First, recall that if P is a peak set for X, then no arc of $E_{[\geq X]}$ leaves P by definition. This implies that no undominated arc leaves P, and therefore, each strongly connected component of $D^H[U]$ is either entirely contained in P, or is disjoint from it. Hence, if $\mathcal{S}$ denotes the family of strongly connected components in $D^H[U]$, then $P = \bigcup_{S \in \mathcal{S}_P} S$ for some $\mathcal{S}_P \subseteq \mathcal{S}$.

By Lemma 3.4, the arc $(p, X(p))$ is undominated for each $p \in P$. Since no undominated arc leaves P, this implies that X contains a collection of pairwise vertex-disjoint cycles in U covering all agents in P. By the definition of strongly connected components, each such cycle must be entirely contained in some strongly connected component of $D^H[U]$. Summarizing this, we obtain that for each $S \in \mathcal{S}_P$ the arc set $X_S = \{(s, X(s)) : s \in S\}$ is an allocation in $H_{|S}$ that is contained in U and disjoint from F. Consequently, each set $S \in \mathcal{S}_P$ is added to $\mathcal{T}$ on line 9 of Algorithm SCFA. Hence, at the beginning of the iteration on lines 10–23, $P \subseteq \bigcup_{T \in \mathcal{T}} T$ holds.

We prove that this remains true throughout the run of the algorithm. For this, it suffices to see that whenever some set $S \in \mathcal{S}$ is removed from T, then $S \notin \mathcal{S}_P$. So assume that $S \in \mathcal{S}_P$ is removed from $\mathcal{T}$ on line 23, and before the removal, $P \subseteq T^\star$ for $T^\star = \bigcup_{T \in \mathcal{T}} T$. Then $H_{|S}$ does not admit an allocation in $E_{S,\mathcal{T}}$, that is, $H_{|S}$ does not admit a $T^\star$-valid allocation.

Recall now that allocation X_S for $H_{|S}$ is P-valid due to Lemma 3.4. However, since $P \subseteq T^\star$, all arcs leaving $T^\star$ necessarily leave P as well, and hence must be dominated by an arc of X_S. It follows that X_S is $T^\star$-valid. This contradicts our assumption that $H_{|S}$ does not admit a $T^\star$-valid allocation, proving that no set in $\mathcal{S}_P$ is ever removed by the algorithm from $\mathcal{T}$. $\square$

The correctness of Algorithm SCFA is stated below.

THEOREM 4.4 ($\star$). *Algorithm SCFA correctly solves each instance of SCFA, and runs in $O(|N|^2 \cdot |E|^{1+o(1)})$ time where (N,E) is the underlying graph of the housing market H of the instance.*

PROOF SKETCH. The proof applies induction on the number of agents; the correctness is clear for $|N| = 1$.

We need to show that whenever Algorithm SCFA concludes that the instance does not admit a solution, then this indeed holds. First, if a solution exists, then the algorithm never reaches line 24 as the peak set of the solution remains in $\bigcup_{T \in \mathcal{T}} T$ by Lemma 4.3. Second, Algorithm SCFA cannot return $\varnothing$ on line 21 either if a solution exists, due to the induction hypothesis.

For the other direction, we need to prove that whenever the algorithm returns a set X of arcs, then $X \in \mathcal{SC}(H)$. This can be shown using Theorem 3.5 in combination with our induction hypothesis.

By $|\mathcal{T}| \leq |N|$, lines 10–23 are performed at most $|N|$ times. The bottleneck in each iteration is to compute a perfect matching in a bipartite graph, which takes $O(|E|^{1+o(1)})$ time [15]. The recursion yields an additional factor of $|N|$ in the running time. $\triangle$

COROLLARY 4.5. *SCFA with n agents and m arcs in the underlying graph can be solved in $O(n^2 \cdot m^{1+o(1)})$ time.*

4.3 Extension to Forbidden and Forced Arcs

We remark that Algorithm SCFA can also be used to solve the extension of SCFA with *forced* arcs.

STRONG CORE WITH FORBIDDEN AND FORCED ARCS (SCFFA):
Input: A housing market H with partial orders, a set F^+ of *forced* arcs and a set F^- of *forbidden* arcs in the underlying directed graph D^H.
Task: Decide if there exists an allocation $X \in \mathcal{SC}(H)$ such that $F^+ \subseteq X$ and $X \cap F^- = \emptyset$, and if so, find such an allocation.

To solve SCFFA in a housing market H, it is sufficient to solve the variant that only involves forbidden arcs. Indeed, since an allocation contains *exactly one* outgoing arc for each agent, we know that an allocation in H contains an arc (a, b) of the underlying graph D^H if and only if it does not contain any of the arcs $\delta^{\text{out}}(a) \setminus \{(a, b)\}$ in D^H. Therefore, we can reduce an instance of (H, F^+, F^-) of SCFFA to an equivalent instance (H, F) of SCFA where $F = F^- \cup \bigcup_{(a,b) \in F^+} \left(\delta_A^{\text{out}}(a) \setminus \{(a, b)\} \right)$. Thus, we obtain the following consequence of Corollary 4.5.

COROLLARY 4.6. *SCFFA with n agents and m arcs in the underlying graph can be solved in $O(n^2 \cdot m^{1+o(1)})$ time.*

4.4 Strategyproofness

A desirable property of Algorithm SCFA is that it is *weak group-strategyproof*, i.e., it is not possible for a coalition of agents to strictly improve their situation via misrepresenting their preferences.

To formalize this definition, let $f_{\text{SCFA}}(H, F)$ denote the set of allocations that can be obtained via Algorithm SCFA on input (H, F); note that there can be several such allocations due to line 15, where the algorithm may pick an arbitrary one among a set of possible allocations for a given submarket $H_{|S}$. Given a housing market $H = (N, \{>_a : a \in N\})$ and a coalition $C \subseteq N$ of agents, a C-*deviation* from H is a housing market $H' = (N, \{>'_a : a \in N\})$ where $>_a = >'_a$ for all $a \in N \setminus C$.

Definition 4.7. Let f be a mechanism that assigns to each instance (H, F) of the STRONG CORE WITH FORBIDDEN ARCS problem a set $f(H, F) \subseteq \mathcal{SC}(H)$ of solutions for (H, F). We say that f is *weakly group-strategyproof* (with respect to SCFA) if there does not exist an instance (H, F) with $H = (N, \{>_a : a \in N\})$, a coalition $C \subseteq N$ and a C-deviation H' from H such that there are allocations $X \in f(H, F)$ and $X' \in f(H', F)$ for which $X'(c) >_c X(c)$ for each $c \in C$.

We remark that Definition 4.7 takes into account models deviations where agents may only misreport their preferences but have no possibility to change the set of forbidden arcs.

THEOREM 4.8 (⋆). *Algorithm SCFA is weakly group-strategyproof.*

PROOF SKETCH. The proof relies on the observation that, roughly speaking, whenever an agent p is removed from the market during Algorithm SCFA, the allocation fixed for p is the best p can hope for, as long as no agent removed earlier than (or together with) p is contained in the deviating coalition C. Using this claim for the first agent c of C removed from the market, one can show that there is at least one agent removed before c for which the allocation obtained as a result of the deviation is disadvantageous; this gives rise to a blocking cycle, leading to a contradiction that proves the result. ⊿

4.5 Finding All Allocations in the Strong Core

It is known that the Quint–Wako algorithm can find all allocations in the strong core of a housing market $H = (N, \{>_a\}_{a \in N})$ in which agents' preferences are described by weak orders. More precisely, if S is an absorbing set of the graph D^H underlying H, then by choosing an allocation (consisting of undominated arcs) for the submarket $H_{|S}$ appropriately, and using also appropriate choices

within the recursive call on $H_{|(N \setminus S)}$, the Quint–Wako algorithm can return any fixed allocation X in the strong core of H.

Interestingly, this also holds for Algorithm SCFA if agents' preferences are weak orders; hence, Algorithm SCFA is as powerful as the Quint–Wako algorithm when used for enumerating the strong core. By contrast, if agents' preferences are partial orders, then the strong core may contain allocations that can never be returned by Algorithm SCFA. For details and examples, see the full version.

5 PROPERTIES OF THE STRONG CORE

In previous sections, we already mentioned that the strong core may be empty when we assume that agents' preferences are partial orders. If the strong core is not empty, then we are interested in knowing some of its most relevant characteristics.

Incomparability of strong core allocations. We start with the remarkable fact that all agents find any two allocations X and Y in the strong core incomparable. This is stated in Lemma 5.1 whose proof relies on structural observations about the arc set obtained, roughly speaking, by letting each agent a choose between $(a, X(a))$ and $(a, Y(a))$ according to its preferences.

LEMMA 5.1 (⋆). *Given a housing market $H = (N, \{>_a : a \in N\})$ with partial order preferences, and allocations X and Y in the strong core of H, we have $X \sim_a Y$ for each agent $a \in N$.*

An important consequence of Lemma 5.1 is that even though there might be allocations in the strong core that can never be found by Algorithm SCFA, this is not really relevant from the viewpoint of the agents, as all agents find any two allocations in the strong core incomparable. Lemma 5.1 also plays a crucial role in proving that the strong core respects improvement.
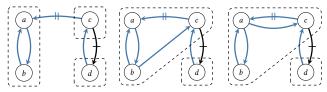
The effect of improvements on the strong core. We next aim to examine the effect of a change in the market when the house of some agent p improves—is such a change necessarily (weakly) advantageous for p in terms of the strong core? Such questions are essential to understand the incentives of agents to improve their endowments. In the context of a kidney exchange program, does bringing a "better" (e.g., younger) donor always benefits the patient, assuming that the program offers a strong core allocation?

To formalize the concept of improvement, consider two housing markets $H = (N, \{>_a : a \in N\})$ and $H' = (N, \{>'_a : a \in N\})$ over the same agent set N. For two agents $p, q \in N$, we say that H' is a (p, q)-*improvement* of H if H' is obtained from H by shifting the position of p "upward" in the preferences of agent q; see also Example 5.2. Formally, we require that

(1) for each $a \in N \setminus \{q\}$, $>_a = >'_a$, that is, we only allow changes in the preferences of agent q;

(2) for each $a \in N \setminus \{p\}$, $p >_q a$ implies $p >'_q a$, and $p \geq_q a$ implies $p \geq'_q a$, that is, agent p may only become more preferred for agent q in $>'$, and not less preferred;

(3) for each $a, b \in N \setminus \{p\}$, we have $a >_q b$ if and only if $a >'_q b$, that is, the preferences for agent q remain unchanged when comparing two agents not involving p.

Now, a p-*improvement* of H is the result of a sequence of (p, q_i)-improvements from H for some series of agents $q_1, \ldots, q_k \in N$.

Example 5.2. Consider the following market H^3 over agent set $N = \{a, b, c, d\}$; see Figure 3a. The acceptability sets are $A(a) =$

(a) Original housing market H^3.

(b) Market H^4: a (c, b)-improvement of H^3.

(c) Market H^5, a (c, a)-improvement of H^3.

Figure 3: Illustration for Example 5.2.

$A(b) = \{a, b\}$, $A(c) = \{a, c, d\}$, and $A(d) = \{c, d\}$. For each agent x and $y \in A(x) \setminus \{x\}$, we let $y >_x x$; additionally, we set $a >_c d$. Then $\mathcal{SC}(H^3) = \{X_{\text{orig}}\}$ where $X_{\text{orig}} = \{(a, b), (b, a), (c, c), (d, d)\}$.

Let H^4 be the c-improvement of H^3 where the preferences are defined the same way as in H^3 but b now prefers c to its own house, so its acceptability set becomes $A(b) = N \setminus \{d\}$; see Figure 3b. Then $\mathcal{SC}(H^4)$ contains the unique allocation $\{(a, b), (b, c), (c, a), (d, d)\}$ which c prefers to X_{orig}.

Let now H^5 be the c-improvement of H^3 where the preferences are defined the same way as in H^3 but a now prefers c to its own house; see Figure 3c. The strong core of H^5 then becomes empty, since the strongly connected component $\{a, b, c\}$ within the subgraph of undominated arcs does not contain an allocation.

To study the effects of change in a housing market, we investigate how the strong core changes when a p-improvement occurs for some agent $p \in N$. Particularly, we aim to understand whether a p-improvement is always beneficial (or, at least, not detrimental) to agent p; hence, we aim to check the following property.

Definition 5.3. The strong core of housing markets *respects improvement*, if for all housing markets H and H' such that H' is a p-improvement of H for some agent p, we have $X' \succeq_p X$ for all allocations $X \in \mathcal{SC}(H, p)$ and $X' \in \mathcal{SC}(H', p)$.

Note that Definition 5.3 allows the strong core of H' to become empty as the result of an improvement, which may indeed happen if agents' preferences are weak or partial orders, as we have already seen in Example 5.2.

Remark 2. Notably, Lemma 5.1 does not hold for the core of a housing market with indifferences. Hence, Biró et al. [11] introduced the following notion: the core of housing markets *respects improvement for the best available house* if for all housing markets H and H' such that H' is a p-improvement of H for some agent p, $X' \succeq_p X$ holds whenever X and X' are among the most-preferred allocations by p within the core of H and of H', respectively; the respecting improvement property for the *worst* available house is defined analogously. These two notions coincide for the strong core with the property expressed in Definition 5.3. Schlotter et al. [41] proved that the core of housing markets respects improvement for the best available house, but not for the worst available house.

Table 2 summarizes the results regarding the property of respecting improvements for the strong core and the core, including Theorem 5.4, the main result of this section. Although the proof of Theorem 5.4 is elementary in its technique, it requires certain insights and uses ideas from the proof of Lemma 5.1.

THEOREM 5.4 (⋆). *Let $H = (N, \{>_a : a \in N\})$ be a housing market in which agents' preferences are partial orders, and let H' be a p-improvement of H for some agent p. Then for all allocations X and X' in the strong core of H and H', respectively, it holds that $X' \succeq_p X$.*

Table 2: Summary of results on the effect of improvements. RI stands for "respects improvement", while RI-best / RI-worst stands for "respects improvement for the best / worst available house", respectively. We use ✔ to indicate that the given property holds; ✔ ∃ additionally signals that the strong core (or core) cannot become empty as the result of a p-improvement; ✗ means that the given property fails to hold.

Preferences	Strong core RI	Core RI-best	Core RI-worst
Strict orders	✔ ∃ [11]	✔ ∃ [41]	✗ [41]
Weak orders	✔ [11]	✔ ∃ [41]	✗ [41]
Partial orders	✔ Thm. 5.4	✔ ∃ [41]	✗ [41]

Formulation as an integer linear program. Quint and Wako were the first to provide an ILP for the strong core of housing markets [33]. Subsequently, Biró, Klijn, Klimentova, and Viana [11] provided an improved ILP formulation for the strong core where the number of constraints is only linear in the size of the market. Biró et al. [11] assumed weakly ordered preferences, hence it is not immediately clear whether their formulation remains correct for partial order preferences. We prove that the ILP formulation for the strong core by Biró et al. remains sound even when agents' preferences are partial orders; see the full version for the details.

6 CONCLUSION

We investigated the strong core of housing markets in a model where agents' preferences are partial orders, filling a gap in the literature. Our main result is a polynomial-time algorithm for obtaining an allocation in the strong core whenever the strong core is not empty; our algorithm is weakly group-strategyproof and can accommodate forbidden and forced arcs. We also discovered important properties of the strong core in housing markets with partial order preferences. In particular, we showed that the strong core in such housing markets respects improvements, and that the ILP proposed by Biró et al. [11] for the strong core remains sound even when agents' preferences are partial orders.

As the strong core can be empty in housing markets with indifferences, it would be interesting to study allocations that are "nearly in the strong core". E.g., can we find allocations for which a few agents cover all blocking cycles? Such an allocation could be stabilized by compensating these agents, thus preventing deviations. Or can we find an allocation from which no deviation yields a strict improvement for at least k agents for some constant $k \in \mathbb{N}$?

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